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# Discrete-Time Synergetic Optimal Control of Nonlinear Systems

Nusawardhana,* S. H. Żak,† and W. A. Crossley‡
*Purdue University, West Lafayette, Indiana 47907*

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**Discrete-time synergetic optimal control strategies are proposed for a class of nonlinear discrete-time dynamic systems, in which a special performance index is used that results in closed-form solutions of the optimal problems. Reduced-dimension aggregated variables representing a combination of the actual controlled plant variables are used to define the performance index. The control law optimizing the performance index for the respective nonlinear dynamic system is derived from the associated first-order difference equation in terms of the aggregated variables. Some connections between discrete-time synergetic control and the discrete-time linear quadratic regulator as well as discrete-time variable-structure sliding-mode controls are established. A control design procedure leading to closed-loop stability of a class of nonlinear systems with matched nonlinearities is presented. For these types of systems, the discrete-time synergetic optimal control strategy for tracking problems is developed by incorporating integral action. The closed-loop stability depends upon proper construction of the aggregated variables so that the closed-loop nonlinear system on the manifold specified by the aggregated variables is asymptotically stable. An algorithm for the construction of such a stabilizing manifold is given. Finally, the results are illustrated with an example that involves a discrete-time nonlinear helicopter model.**

## Nomenclature

| | | |
|---|---|---|
| $\boldsymbol{A}$ | = | state matrix in $\mathbb{R}^{n\times n}$ |
| $\tilde{\boldsymbol{A}}$ | = | state matrix of the augmented state-space model |
| $\boldsymbol{A}(\boldsymbol{x})$ | = | state-dependent vector function of dimension $n$ |
| $\boldsymbol{A}^x$, $\boldsymbol{A}^w$, $\boldsymbol{A}^z$ | = | state matrices associated with state vectors $\boldsymbol{x}$, $\boldsymbol{w}$, $\boldsymbol{z}$ |
| $a_i$ | = | the $i$th parameter appearing in the helicopter dynamic model |
| $\boldsymbol{B}$ | = | input matrix in $\mathbb{R}^{n\times m}$ |
| $\tilde{\boldsymbol{B}}$ | = | input matrix of the augmented state-space model |
| $\boldsymbol{B}(\boldsymbol{x})$ | = | state-dependent input matrix function of size $n\times m$ |
| $\boldsymbol{B}_u$, $\boldsymbol{B}_f$ | = | input matrix with respect to $\boldsymbol{u}$ and $\boldsymbol{f}$ inputs |
| $\boldsymbol{F}$ | = | gain matrix of state-feedback control |
| $\boldsymbol{f}$ | = | state-dependent vector of nonlinear function of dimension $q$ |
| $h$ | = | altitude variable |
| $\boldsymbol{I}$, $\boldsymbol{I}_m$ | = | identity matrix, identity matrix in $\mathbb{R}^{m\times m}$ |
| $J$ | = | numerical value of the cost functional in optimal control problems |
| $\boldsymbol{j}$ | = | imaginary number operator, $\sqrt{-1}$ |
| $k$ | = | discrete time |
| $k_0$, $k_f$ | = | initial and final time in the discrete-time instance |
| $L(\cdot)$ | = | cost-functional integrand |
| $\mathcal{M}$ | = | manifold $\{\boldsymbol{x}{:}\boldsymbol{\sigma}(\boldsymbol{x})=\boldsymbol{0}\}$ of dimension $n-m$ |
| $\boldsymbol{N}$ | = | weight matrix for the cross term between state and control variables in the linear quadratic regulator integrand |
| $\boldsymbol{O}$ | = | zero matrix |
| $\boldsymbol{P}$, $\tilde{\boldsymbol{P}}$ | = | matrices, solutions to the Riccati equation or the algebraic Hamilton–Jacobi–Bellman equation |
| $\boldsymbol{Q}$ | = | weight matrix for state variables in the linear quadratic regulator integrand |
| $\boldsymbol{R}$ | = | weight matrix for control variables in the linear quadratic regulator integrand |
| $\boldsymbol{r}$ | = | vector of reference variables of dimension $p$ |
| $\boldsymbol{S}$ | = | surface/hyperplane matrix of dimension $m\times n$ |
| $\boldsymbol{T}$ | = | symmetric positive-definite constant matrix, design parameter matrix, $\boldsymbol{T}>0$ |
| $\boldsymbol{u}$ | = | input vector of dimension $m$ |
| $\boldsymbol{V}$ | = | matrix that completes the column rank of $\boldsymbol{B}$ |
| $\boldsymbol{v}$ | = | input derivative vector or auxiliary control vector in helicopter model dynamics |
| $\boldsymbol{W}^g$ | = | matrix orthogonal to $\boldsymbol{B}$, a left inverse of $\boldsymbol{W}$, so that $\boldsymbol{W}^g\boldsymbol{W}=\boldsymbol{I}_{n-m}$ |
| $\boldsymbol{x}$ | = | state-variable vector of dimension $n$ |
| $\boldsymbol{y}$ | = | output-variable vector of dimension $p$ |
| $\boldsymbol{z}$ | = | state-variable vector of dimension $n$ |
| $\mathcal{Z}$, $\mathcal{Z}^{-1}$ | = | one-step-ahead and one-step-delay matrix operators |
| $\boldsymbol{0}$ | = | zero column vector |
| $\beta$ | = | variant of discrete-time sliding-mode reaching law |
| $\Gamma$ | = | design parameter in the discrete-time tracking problem |
| $\Delta$ | = | difference operator |
| $\zeta$ | = | temporary variable in the helicopter dynamic model |
| $\theta_c$ | = | collective pitch angle |
| $\boldsymbol{\Lambda}$ | = | solution to the polynomial matrix equation |
| $\boldsymbol{\Lambda}^+$, $\boldsymbol{\Lambda}^-$ | = | additive solution and subtractive solution to the polynomial matrix equation |
| $\lambda$ | = | poles or eigenvalues |
| $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}(\boldsymbol{x})$ | = | vector in $\mathbb{R}^m$, an aggregated variable of $\boldsymbol{x}$ defining the manifold $\mathcal{M}$ |
| $\tau$ | = | sampling time |
| $\omega$ | = | angular velocity |

## I. Introduction

OPTIMAL control of a discrete-time dynamic system is concerned with finding a sequence of decisions over discrete time to optimize a given objective functional. Dynamic programming (DP) offers a computational procedure based on the principle of optimality to determine the optimizing control sequence [1–6]. The optimizing control sequence in discrete-time optimal

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*Doctoral Graduate, School of Aeronautics and Astronautics; currently Cummins Inc., Columbus, IN 47203. Member AIAA.

†Professor, School of Electrical and Computer Engineering, Electrical Engineering Building, 465 Northwestern Avenue.

‡Associate Professor, School of Aeronautics and Astronautics, 701 West Stadium Avenue. Associate Fellow AIAA.

control problems can be obtained by solving the Bellman equation. Even though DP offers a systematic procedure to compute the optimizing sequence, the computational requirements to complete this task may become prohibitively large [2,7]. One way to overcome the excessive computational complexity is to use methods that generate efficiently approximate solutions [7].

Closed-form analytical solutions to optimal control problems of discrete-time dynamic systems are attainable for linear systems with quadratic objective functionals [8–12]. Different forms of objective functionals lead to different control laws [13,14]. In this paper, we consider a special form of the objective functional that allows us to obtain closed-form analytical solutions to optimal control problems for a class of discrete-time nonlinear dynamic systems. The format of such an objective functional resembles that of the optimal control problem for continuous-time dynamic systems, leading to the so-called synergetic control strategies. The synergetic control method was recently developed by Kolesnikov et al. [15–17] and applied to engineering energy-conversion problems [18,19]. Nusawardhana et al. [20] also present a list of references on synergetic control, including those of Kolesnikov et al. [15–17].

The synergetic control approach yields an optimizing control law that is derived from the associated linear first-order ordinary differential equation rather than a nonlinear partial differential equation as in the classical continuous optimal control. In [20–22], an alternative derivation of synergetic optimal controls, from both necessary and sufficient optimality conditions, is offered and the proposed controllers are applied to aerospace problems.

The term *synergetics*, as described by Haken in [23–26], is concerned with the spontaneous formation of macroscopic spatial, temporal, or functional structures of systems via self-organization and is associated with systems composed of many subsystems, which may be of quite different natures. One important subject of discussion in these references is the cooperation of these subsystems on macroscopic scales.

The synergetics principle translates to the field of control in terms of state-variable aggregation. As an illustration, consider a dynamic system with an $n$-dimensional state vector $\boldsymbol{x}$ and an $m$-dimensional input variable. Let $\boldsymbol{\sigma}$ be a variable defined as $\boldsymbol{\sigma} = \boldsymbol{S}\boldsymbol{x}$, where $\boldsymbol{S}$ is a matrix in $\mathbb{R}^{m\times n}$and constitutes an aggregated or metavariable. Aggregated variables constitute macroscopic variables from the perspective of synergetics. To be consistent with one of the control objectives, the selection or the construction of macroscopic variables must avoid internal divergence. In other words, stability observed at the macroscopic level implies stability at the microscopic level.

The fundamental concept in the synergetic control approach is that of an invariant manifold and the associated law governing the controlled system's trajectory convergence to the invariant manifold. In this paper, we show that synergetic control is an effective means of controlling discrete-time nonlinear systems. We add that discrete-time dynamics commonly arise after time discretization of the original continuous-time system is performed to digitally simulate and/or control its dynamic behavior. However, there are dynamic systems that are inherently discrete-time. Economic systems are often represented in the discrete-time domain because control strategies for these systems are applied in discrete time (see, for example, [27] and references therein).

The first objective of our presentation here is to present our derivation of the discrete-time synergetic control from both discrete-time calculus of variations and the discrete-time Bellman equation (or dynamic programming) and to establish some connections with the discrete-time variable-structure sliding-mode approach. We show that discrete-time synergetic control not only satisfies the necessary condition for optimality but also satisfies the sufficient condition for optimality. We show that the discrete-time synergetic control for linear systems is the same as the discrete-time nonlinear variable-structure sliding-mode controller employing a certain reaching law. The second objective is to show that for linear time-invariant (LTI) systems and for a special form of the performance index, the synergetic control approach yields the same controller structure as the linear quadratic regulator (LQR) approach. The third objective is to present our stability analysis of the discrete-time synergetic control and a constructive algorithm to design a stabilizing manifold. The fourth objective is to present a discrete-time synergetic control-based design procedure for tracking a reference command using an integral action. The integral action is commonly used in the design of controllers to track constant or almost-constant reference signals. Stability analysis of the closed-loop system is performed. The final objective is to apply the proposed method to a tracking problem of a laboratory-scale helicopter model.

## II. Plant Model and Problem Statement

We consider a class of nonlinear discrete-time dynamic systems modeled by

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}(\boldsymbol{x}_k) + \boldsymbol{B}(\boldsymbol{x}_k)\boldsymbol{u}_k, \tag{1}$$

where $\boldsymbol{x}_k \in \mathbb{R}^n$ is a vector of state variables, $\boldsymbol{u}_k \in \mathbb{R}^m$ is an input vector, $\boldsymbol{A}(\boldsymbol{x})$ is a state-dependent vector function of dimension $n$, and $\boldsymbol{B}(\boldsymbol{x})$ is an $n \times m$ state-dependent input matrix. We define a macrovariable

$$\boldsymbol{\sigma}_k = \boldsymbol{\sigma}(\boldsymbol{x}_k) = \boldsymbol{S}\boldsymbol{x}_k \tag{2}$$

where the constant matrix $\boldsymbol{S} \in \mathbb{R}^{m\times n}$ is constructed so that the square $m \times m$ matrix $\boldsymbol{S}\boldsymbol{B}(\boldsymbol{x}_k)$ is invertible. We wish to solve the following optimal control problem. For a given class of nonlinear systems modeled by Eq. (1), construct a control sequence $\boldsymbol{u}_k$ $[k \in [k_0, \infty)]$ that minimizes the performance index:

$$\begin{aligned} J &= \sum_{k=k_0}^{\infty} \boldsymbol{\sigma}_k^\top \boldsymbol{\sigma}_k + \Delta\boldsymbol{\sigma}_k^\top \boldsymbol{T} \Delta\boldsymbol{\sigma}_k \\ &= \sum_{k=k_0}^{\infty} \boldsymbol{\sigma}_k^\top \boldsymbol{\sigma}_k + (\boldsymbol{\sigma}_{k+1} - \boldsymbol{\sigma}_k)^\top \boldsymbol{T} (\boldsymbol{\sigma}_{k+1} - \boldsymbol{\sigma}_k) \end{aligned} \tag{3}$$

where an $m \times m$ symmetric positive-definite matrix $\boldsymbol{T} = \boldsymbol{T}^\top > 0$ is a design parameter matrix. The performance index $J$ depends on $\boldsymbol{u}_k$ through the variable $\boldsymbol{\sigma}_{k+1}$. Note that $\boldsymbol{\sigma}_{k+1} = \boldsymbol{\sigma}(\boldsymbol{x}_{k+1})$ depends upon $\boldsymbol{u}_k$ because

$$\boldsymbol{\sigma}_{k+1} = \boldsymbol{\sigma}(\boldsymbol{x}_{k+1}) = \boldsymbol{\sigma}(\boldsymbol{A}(\boldsymbol{x}_k) + \boldsymbol{B}(\boldsymbol{x}_k)\boldsymbol{u}_k)$$

The performance index (3) has not been commonly used in the discrete-time optimal control literature. However, this novel performance index leads to new types of optimal control strategies that are expressible in a closed form, as we demonstrate in this paper.

## III. Controller Construction Using the Calculus of Variations

Our derivation of the discrete-time synergetic control law is rooted in the calculus of variations. Continuous functional optimization in the calculus of variations is solved using the continuous Euler–Lagrange equation. The well-known fundamental principle of the calculus of variations states that for a functionals to be extremized, the first variation of the functional must be equal to zero (see, for example, Theorem I on page 178 in [28], Theorem 3.1 on page 7 in [29], or pages 35–38 in [30]). In the discrete-time case, we have the following theorem for the functional extremum which is adapted from page 127 in [31] (see also Sec. 5.1.1 of [32]). Discussion and mathematical derivation of the discrete-time Euler–Lagrange equation in these references serve as the proof to the following theorem.

*Theorem 1 (a necessary condition for extremum):* Let

$$J = \sum_{k=k_0}^{k_f - 1} L(\boldsymbol{x}_k, \boldsymbol{x}_{k+1}, k)$$

be a functional defined on the set of functions $\boldsymbol{x}_k$ that have continuous first derivatives in $[k_0, k_f]$ and satisfy boundary conditions $\boldsymbol{x} = \boldsymbol{x}_{k_0}$ and $\boldsymbol{x}_f = \boldsymbol{x}_{k_f}$. Then the necessary condition for $\boldsymbol{x}_k$ to be an extremizer of the cost $J$ is that $\boldsymbol{x}_k$ satisfies the difference equation:

$$\frac{\partial}{\partial \boldsymbol{x}_k} L(\boldsymbol{x}_k, \boldsymbol{x}_{k+1}, k) + \frac{\partial}{\partial \boldsymbol{x}_k} L(\boldsymbol{x}_{k-1}, \boldsymbol{x}_k, k-1) = \boldsymbol{0} \tag{4}$$

where the derivative operator $(\partial/\partial \boldsymbol{x}_k)(\cdot)$, applied to a real-valued function of many variables, is a column vector.

We now apply the preceding necessary condition to the cost functional (3) to obtain

$$\frac{\partial}{\partial \boldsymbol{\sigma}_k}\{\boldsymbol{\sigma}_k^\top \boldsymbol{\sigma}_k + (\boldsymbol{\sigma}_{k+1} - \boldsymbol{\sigma}_k)^\top \boldsymbol{T}(\boldsymbol{\sigma}_{k+1} - \boldsymbol{\sigma}_k)\} + \frac{\partial}{\partial \boldsymbol{\sigma}_k}\{\boldsymbol{\sigma}_{k-1}^\top \boldsymbol{\sigma}_{k-1} + (\boldsymbol{\sigma}_k - \boldsymbol{\sigma}_{k-1})^\top \boldsymbol{T}(\boldsymbol{\sigma}_k - \boldsymbol{\sigma}_{k-1})\} = \boldsymbol{0} \tag{5}$$

Performing simple manipulations yields

$$\begin{aligned} &2\boldsymbol{\sigma}_k - 2\boldsymbol{T}(\boldsymbol{\sigma}_{k+1} - \boldsymbol{\sigma}_k) + 2\boldsymbol{T}(\boldsymbol{\sigma}_k - \boldsymbol{\sigma}_{k-1}) \\ &\quad = 2((\boldsymbol{I} + 2\boldsymbol{T})\boldsymbol{\sigma}_k - \boldsymbol{T}\boldsymbol{\sigma}_{k+1} - \boldsymbol{T}\boldsymbol{\sigma}_{k-1}) = \boldsymbol{0} \end{aligned} \tag{6}$$

Rearranging terms in Eq. (6) and dividing both sides by 2, we obtain

$$\boldsymbol{T}\boldsymbol{\sigma}_{k+1} - (\boldsymbol{I} + 2\boldsymbol{T})\boldsymbol{\sigma}_k + \boldsymbol{T}\boldsymbol{\sigma}_{k-1} = \boldsymbol{0} \tag{7}$$

Let $\mathcal{Z}$ be a one-step-ahead operator defined as

$$\mathcal{Z}\boldsymbol{\sigma}_k = \boldsymbol{\sigma}_{k+1}$$

then $\mathcal{Z}^{-1}\boldsymbol{\sigma}_k = \boldsymbol{\sigma}_{k-1}$. Using the operator $\mathcal{Z}$, the difference equation (7) can be rewritten as

$$(\boldsymbol{T}\mathcal{Z} - (\boldsymbol{I} + 2\boldsymbol{T}) + \boldsymbol{T}\mathcal{Z}^{-1})\boldsymbol{\sigma}_k = \boldsymbol{0} \tag{8}$$

For Eq. (8) to hold for all values of $\boldsymbol{\sigma}_k$, the following matrix equation must also hold:

$$\boldsymbol{I}\mathcal{Z} - \boldsymbol{T}^{-1}(\boldsymbol{I} + 2\boldsymbol{T}) + \boldsymbol{I}\mathcal{Z}^{-1} = \boldsymbol{O} \tag{9}$$

where $\boldsymbol{O}$ denotes a zero matrix. Postmultiplying Eq. (9) by $\mathcal{Z}$, we obtain

$$\boldsymbol{I}\mathcal{Z}^2 - \boldsymbol{T}^{-1}(\boldsymbol{I} + 2\boldsymbol{T})\mathcal{Z} + \boldsymbol{I} = \boldsymbol{O} \tag{10}$$

This is a quadratic matrix equation that can be represented in a factored form:

$$(\boldsymbol{I}\mathcal{Z} - \boldsymbol{\Lambda}^+)(\boldsymbol{I}\mathcal{Z} - \boldsymbol{\Lambda}^-) = \boldsymbol{O} \tag{11}$$

where applying the matrix version of the quadratic formula gives

$$\begin{aligned} \boldsymbol{\Lambda}^+ &= \tfrac{1}{2}\boldsymbol{T}^{-1}(\boldsymbol{I} + 2\boldsymbol{T}) + \tfrac{1}{2}(\boldsymbol{T}^{-2}(\boldsymbol{I} + 2\boldsymbol{T})^2 - 4\boldsymbol{I})^{\frac{1}{2}} \\ \boldsymbol{\Lambda}^- &= \tfrac{1}{2}\boldsymbol{T}^{-1}(\boldsymbol{I} + 2\boldsymbol{T}) - \tfrac{1}{2}(\boldsymbol{T}^{-2}(\boldsymbol{I} + 2\boldsymbol{T})^2 - 4\boldsymbol{I})^{\frac{1}{2}} \end{aligned}$$

To ensure the stability of the closed-loop discrete-time system, it is desirable to obtain a stabilizing solution to Eq. (10). We choose $\boldsymbol{\Lambda}^-$ as the solution to the polynomial matrix Eq. (10) because all eigenvalues of $\boldsymbol{\Lambda}^-$ lie inside the unit disc. To see this, we perform manipulations:

$$\begin{aligned} \boldsymbol{\Lambda}^- &= \tfrac{1}{2}\boldsymbol{T}^{-1}(\boldsymbol{I} + 2\boldsymbol{T}) - \tfrac{1}{2}(\boldsymbol{T}^{-2}(\boldsymbol{I} + 2\boldsymbol{T})^2 - 4\boldsymbol{I})^{\frac{1}{2}} \\ &= \tfrac{1}{2}\boldsymbol{T}^{-1}(\boldsymbol{I} + 2\boldsymbol{T}) - \tfrac{1}{2}(\boldsymbol{T}^{-2}(\boldsymbol{I} + 4\boldsymbol{T} + 4\boldsymbol{T}^2) - 4\boldsymbol{I})^{\frac{1}{2}} \\ &= \tfrac{1}{2}\boldsymbol{T}^{-1}(\boldsymbol{I} + 2\boldsymbol{T}) - \tfrac{1}{2}(\boldsymbol{T}^{-2}(\boldsymbol{I} + 4\boldsymbol{T}) + 4\boldsymbol{I} - 4\boldsymbol{I})^{\frac{1}{2}} \\ &= \tfrac{1}{2}\boldsymbol{T}^{-1}(\boldsymbol{I} + 2\boldsymbol{T}) - \tfrac{1}{2}(\boldsymbol{T}^{-2}(\boldsymbol{I} + 4\boldsymbol{T}))^{\frac{1}{2}} \\ &= \tfrac{1}{2}\boldsymbol{T}^{-1}(\boldsymbol{I} + 2\boldsymbol{T}) - \tfrac{1}{2}\boldsymbol{T}^{-1}(\boldsymbol{I} + 4\boldsymbol{T})^{\frac{1}{2}} < \tfrac{1}{2}\boldsymbol{T}^{-1}(\boldsymbol{I} + 2\boldsymbol{T}) - \tfrac{1}{2}\boldsymbol{T}^{-1} \\ &= \boldsymbol{I}_m \end{aligned}$$

It is easy to verify that $\boldsymbol{\Lambda}^+\boldsymbol{\Lambda}^- = \boldsymbol{I}_m$ and thus $\boldsymbol{\Lambda}^- = (\boldsymbol{\Lambda}^+)^{-1}$. Because $\boldsymbol{\Lambda}^+ > \boldsymbol{O}$, we have $\boldsymbol{\Lambda}^- = (\boldsymbol{\Lambda}^+)^{-1} > \boldsymbol{O}$. Hence, $\boldsymbol{O} < \boldsymbol{\Lambda}^- < \boldsymbol{I}$. This makes $\mathcal{Z}\boldsymbol{I} = \boldsymbol{\Lambda}^-$ the stabilizing solution to the matrix polynomial Eq. (10). Hence, we employ

$$(\mathcal{Z}\boldsymbol{I} - \boldsymbol{\Lambda}^-)\boldsymbol{\sigma}_k = \boldsymbol{0} \Rightarrow \boldsymbol{\sigma}_{k+1} - \boldsymbol{\Lambda}^-\boldsymbol{\sigma}_k = \boldsymbol{0} \tag{12}$$

to derive the extremizing control law.

We summarize the preceding development in the following proposition.

*Proposition 1:* If $\boldsymbol{\sigma}_k$ is a local minimizer of the functional

$$J = \sum_{k=k_0}^{\infty} \boldsymbol{\sigma}_k^\top \boldsymbol{\sigma}_k + (\boldsymbol{\sigma}_{k+1} - \boldsymbol{\sigma}_k)^\top \boldsymbol{T}(\boldsymbol{\sigma}_{k+1} - \boldsymbol{\sigma}_k) \tag{13}$$

on all admissible vector functions of $\boldsymbol{\sigma}_k$, then

$$\boldsymbol{\sigma}_{k+1} - \boldsymbol{\Lambda}\boldsymbol{\sigma}_k = \boldsymbol{0} \tag{14}$$

where

$$\boldsymbol{\Lambda} = \boldsymbol{\Lambda}^- = \tfrac{1}{2}\boldsymbol{T}^{-1}(\boldsymbol{I} + 2\boldsymbol{T}) - \tfrac{1}{2}(\boldsymbol{T}^{-2}(\boldsymbol{I} + 2\boldsymbol{T})^2 - 4\boldsymbol{I})^{\frac{1}{2}}$$

The vector function $\boldsymbol{\sigma}_k$ that satisfies Eq. (14) is referred to as an extremizing $\boldsymbol{\sigma}_k$. Using Proposition 1, we can now state and prove the following theorem.

*Theorem 2 (discrete-time synergetic extremum control):* For a dynamic system model,

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}(\boldsymbol{x}_k) + \boldsymbol{B}(\boldsymbol{x}_k)\boldsymbol{u}_k, \qquad k_0 \le k < \infty, \qquad \boldsymbol{x}_{k_0} = \boldsymbol{x}_0$$

and the associated performance index

$$J = \sum_{k=k_0}^{\infty} \boldsymbol{\sigma}_k^\top \boldsymbol{\sigma}_k + (\boldsymbol{\sigma}_{k+1} - \boldsymbol{\sigma}_k)^\top \boldsymbol{T}(\boldsymbol{\sigma}_{k+1} - \boldsymbol{\sigma}_k)$$

where $\boldsymbol{\sigma}_k = \boldsymbol{S}\boldsymbol{x}_k$, $\boldsymbol{x}_k \in \mathbb{R}^n$, and $\boldsymbol{S}\boldsymbol{B}(\boldsymbol{x}_k)$ is invertible, the extremizing control law is

$$\boldsymbol{u}_k^* = -(\boldsymbol{S}\boldsymbol{B}(\boldsymbol{x}_k))^{-1}(\boldsymbol{S}\boldsymbol{A}(\boldsymbol{x}_k) - \boldsymbol{\Lambda}\boldsymbol{S})\boldsymbol{x}_k \tag{15}$$

*Proof:* By Proposition 1, the trajectory $\boldsymbol{\sigma}_k$ extremizing the performance index (13) must satisfy the difference equation (14). Substituting the system dynamics into $\boldsymbol{\sigma}_{k+1} = \boldsymbol{S}\boldsymbol{x}_{k+1}$ yields

$$\boldsymbol{\sigma}_{k+1} = \boldsymbol{S}(\boldsymbol{A}(\boldsymbol{x}_k) + \boldsymbol{B}(\boldsymbol{x}_k)\boldsymbol{u}_k)$$

Substituting the preceding into the difference equation (14) gives

$$\boldsymbol{S}(\boldsymbol{A}(\boldsymbol{x}_k) + \boldsymbol{B}(\boldsymbol{x}_k)\boldsymbol{u}_k) - \boldsymbol{\Lambda}\boldsymbol{S}\boldsymbol{x}_k = \boldsymbol{0} \tag{16}$$

Solving Eq. (16) for $\boldsymbol{u}_k$, we obtain Eq. (15), which is the control law that extremizes the given performance index. □

Note that we refer to the control law (15) as extremizing rather than minimizing to emphasize the fact that the control law (15) is derived from the necessary optimality condition [33]. However, as we will show in the next section, the control law (15) is also optimizing: in our case, minimizing.

## IV. Derivation of Discrete-Time Synergetic Control from Bellman's Equation

Consider a general discrete-time nonlinear dynamic system modeled by

$$\boldsymbol{x}_{k+1} = \boldsymbol{f}(\boldsymbol{x}_k, \boldsymbol{u}_k), \qquad k_0 \le k < \infty, \qquad \boldsymbol{x}_{k_0} = \boldsymbol{x}_0 \tag{17}$$

where $\boldsymbol{x}_k \in \mathbb{R}^n$. The problem of finding a control sequence $\{\boldsymbol{u}_k : k \in [k_0, \infty)\}$ along with its corresponding state trajectory $\{\boldsymbol{x}_k : k \in [k_0, \infty)\}$ to minimize the performance index

$$J(\boldsymbol{x}_0) = \sum_{k=k_0}^{\infty} L(\boldsymbol{x}_k, \boldsymbol{u}_k) \tag{18}$$

constitutes an infinite-horizon discrete-time optimal control problem. One way to synthesize an optimal feedback control sequence $\boldsymbol{u}_k^*$, $k \ge k_0$ is to employ Bellman's equation [3,8,9], which provides both necessary and sufficient conditions for the optimality of the resulting controller. The following theorem (found in Vol. II, Sec. 1.2, of [3]) states the necessary and sufficient condition for the optimality of the solution to the preceding discrete-time optimal control problem.

*Theorem 3 (necessary and sufficient condition for optimality):* A policy $\boldsymbol{u}_k^*$ is optimal if and only if it is a minimizer of the Bellman equation:

$$\begin{aligned} J^*(\boldsymbol{x}_k) &= \min_{\boldsymbol{u}_k}\{L(\boldsymbol{x}_k, \boldsymbol{u}_k) + J(\boldsymbol{f}(\boldsymbol{x}_k, \boldsymbol{u}_k))\} \\ &= L(\boldsymbol{x}_k, \boldsymbol{u}_k^*) + J^*(\boldsymbol{f}(\boldsymbol{x}_k, \boldsymbol{u}_k^*)) \end{aligned} \tag{19}$$

Using Theorem 3, we construct a synergetic control strategy for a class of nonlinear systems.

*Theorem 4 (discrete-time synergetic optimal control):* Consider a nonlinear system model given by

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}(\boldsymbol{x}_k) + \boldsymbol{B}(\boldsymbol{x}_k)\boldsymbol{u}_k \tag{20}$$

and the associated performance index

$$J(\boldsymbol{\sigma}_{k_0}) = \sum_{k=k_0}^{\infty} L(\boldsymbol{\sigma}_k, \boldsymbol{\sigma}_{k+1}) \tag{21}$$

where

$$L(\boldsymbol{\sigma}_k, \boldsymbol{\sigma}_{k+1}) = \boldsymbol{\sigma}_k^\top \boldsymbol{\sigma}_k + (\boldsymbol{\sigma}_{k+1} - \boldsymbol{\sigma}_k)^\top \boldsymbol{T}(\boldsymbol{\sigma}_{k+1} - \boldsymbol{\sigma}_k) \tag{22}$$

$\boldsymbol{\sigma}_k = \boldsymbol{S}\boldsymbol{x}_k$, $\boldsymbol{x}_k \in \mathbb{R}^n$, and $\boldsymbol{S}\boldsymbol{B}(\boldsymbol{x}_k)$ is invertible. The optimal feedback control law $\boldsymbol{u}_k^*$ that minimizes the performance index (21) is obtained by solving the difference equation:

$$\boldsymbol{\sigma}_{k+1} = (\boldsymbol{T} + \boldsymbol{P})^{-1}\boldsymbol{T}\boldsymbol{\sigma}_k \tag{23}$$

where the symmetric positive-definite $\boldsymbol{P} \in \mathbb{R}^{m\times m}$ is the solution to the algebraic equation:

$$\boldsymbol{P} = \boldsymbol{I} + \boldsymbol{T} - \boldsymbol{T}^\top(\boldsymbol{T} + \boldsymbol{P})^{-1}\boldsymbol{T} \tag{24}$$

The optimal control strategy has the form

$$\boldsymbol{u}_k^* = -((\boldsymbol{T} + \boldsymbol{P})\boldsymbol{S}\boldsymbol{B}(\boldsymbol{x}_k))^{-1}(\boldsymbol{P}\boldsymbol{S}\boldsymbol{A}(\boldsymbol{x}_k) + \boldsymbol{T}\boldsymbol{S}(\boldsymbol{A}(\boldsymbol{x}_k) - \boldsymbol{x}_k)) \tag{25}$$

and the value of the performance index is

$$J^*(\boldsymbol{\sigma}_{k_0}) = \boldsymbol{\sigma}_{k_0}^\top \boldsymbol{P}\boldsymbol{\sigma}_{k_0} \tag{26}$$

where $\boldsymbol{P}$ satisfies Eq. (24).

*Proof:* Let

$$J(\boldsymbol{\sigma}_k) = \boldsymbol{\sigma}_k^\top \boldsymbol{P}\boldsymbol{\sigma}_k \tag{27}$$

The Bellman equation for the optimal control of dynamic system (20) with the performance index (21) takes the form

$$\boldsymbol{\sigma}_k^\top \boldsymbol{P}\boldsymbol{\sigma}_k = \min_{\boldsymbol{u}_k}\{\boldsymbol{\sigma}_k^\top \boldsymbol{\sigma}_k + (\boldsymbol{\sigma}_{k+1} - \boldsymbol{\sigma}_k)^\top \boldsymbol{T}(\boldsymbol{\sigma}_{k+1} - \boldsymbol{\sigma}_k) + \boldsymbol{\sigma}_{k+1}^\top \boldsymbol{P}\boldsymbol{\sigma}_{k+1}\} \tag{28}$$

To find the minimizing sequence $\boldsymbol{u}_k$, we apply the first-order necessary condition for the unconstrained minimization to the expression in the braces in Eq. (28). Note that $\boldsymbol{\sigma}_{k+1}$ depends on $\boldsymbol{u}_k$, but $\boldsymbol{\sigma}_k$ does not. Using the relations

$$\boldsymbol{\sigma}_k = \boldsymbol{S}\boldsymbol{x}_k \quad \text{and} \quad \boldsymbol{\sigma}_{k+1} = \boldsymbol{S}\boldsymbol{x}_{k+1} = \boldsymbol{S}(\boldsymbol{A}(\boldsymbol{x}_k) + \boldsymbol{B}(\boldsymbol{x}_k)\boldsymbol{u}_k)$$

we obtain

$$\begin{aligned} &\frac{\partial}{\partial \boldsymbol{u}_k}\{\boldsymbol{\sigma}_k^\top \boldsymbol{\sigma}_k + (\boldsymbol{\sigma}_{k+1} - \boldsymbol{\sigma}_k)^\top \boldsymbol{T}(\boldsymbol{\sigma}_{k+1} - \boldsymbol{\sigma}_k) + \boldsymbol{\sigma}_{k+1}^\top \boldsymbol{P}\boldsymbol{\sigma}_{k+1}\} \\ &= \frac{\partial}{\partial \boldsymbol{u}_k}\{\boldsymbol{x}_k^\top \boldsymbol{S}^\top \boldsymbol{S}\boldsymbol{x}_k + (\boldsymbol{A}(\boldsymbol{x}_k) + \boldsymbol{B}(\boldsymbol{x}_k)\boldsymbol{u}_k - \boldsymbol{x}_k)^\top \boldsymbol{S}^\top \boldsymbol{T}\boldsymbol{S}(\boldsymbol{A}(\boldsymbol{x}_k) \\ &\quad + \boldsymbol{B}(\boldsymbol{x}_k)\boldsymbol{u}_k - \boldsymbol{x}_k) + (\boldsymbol{A}(\boldsymbol{x}_k) \\ &\quad + \boldsymbol{B}(\boldsymbol{x}_k)\boldsymbol{u}_k)^\top \boldsymbol{S}^\top \boldsymbol{P}\boldsymbol{S}(\boldsymbol{A}(\boldsymbol{x}_k) + \boldsymbol{B}(\boldsymbol{x}_k)\boldsymbol{u}_k)\} \\ &= 2\boldsymbol{B}(\boldsymbol{x}_k)^\top \boldsymbol{S}^\top \boldsymbol{T}\boldsymbol{S}(\boldsymbol{A}(\boldsymbol{x}_k) + \boldsymbol{B}(\boldsymbol{x}_k)\boldsymbol{u}_k - \boldsymbol{x}_k) \\ &\quad + 2\boldsymbol{B}(\boldsymbol{x})^\top \boldsymbol{S}^\top \boldsymbol{P}\boldsymbol{S}(\boldsymbol{A}(\boldsymbol{x}_k) + \boldsymbol{B}(\boldsymbol{x}_k)\boldsymbol{u}_k) \\ &= \boldsymbol{T}\boldsymbol{S}(\boldsymbol{A}(\boldsymbol{x}_k) + \boldsymbol{B}(\boldsymbol{x}_k)\boldsymbol{u}_k) + \boldsymbol{P}\boldsymbol{S}(\boldsymbol{A}(\boldsymbol{x}_k) + \boldsymbol{B}(\boldsymbol{x}_k)\boldsymbol{u}_k) - \boldsymbol{T}\boldsymbol{S}\boldsymbol{x}_k \\ &= (\boldsymbol{T} + \boldsymbol{P})\boldsymbol{S}\boldsymbol{x}_{k+1} - \boldsymbol{T}\boldsymbol{S}\boldsymbol{x}_k = (\boldsymbol{T} + \boldsymbol{P})\boldsymbol{\sigma}_{k+1} - \boldsymbol{T}\boldsymbol{\sigma}_k = \boldsymbol{0} \end{aligned} \tag{29}$$

Therefore, $\boldsymbol{\sigma}_k$, which minimizes the performance index (21), satisfies the first-order difference equation:

$$\boldsymbol{\sigma}_{k+1} - (\boldsymbol{T} + \boldsymbol{P})^{-1}\boldsymbol{T}\boldsymbol{\sigma}_k = \boldsymbol{0}$$

We note that the difference equation (23) yields an asymptotically stable $\boldsymbol{\sigma}$ because all eigenvalues of $(\boldsymbol{T} + \boldsymbol{P})^{-1}\boldsymbol{T}$ lie inside the unit disc. To see this, note that

$$(\boldsymbol{T} + \boldsymbol{P})^{-1}\boldsymbol{T} = (\boldsymbol{I} + \boldsymbol{T}^{-1}\boldsymbol{P})^{-1} < \boldsymbol{I}^{-1} = \boldsymbol{I}$$

We use the difference equation (29) to derive an optimal control law $\boldsymbol{u}_k^*$. To obtain the formula for the optimal control law, we perform the following manipulations. From Eq. (29), we obtain

$$\begin{aligned} (\boldsymbol{T} + \boldsymbol{P})\boldsymbol{\sigma}_{k+1} - \boldsymbol{T}\boldsymbol{\sigma}_k &= (\boldsymbol{T} + \boldsymbol{P})\boldsymbol{S}(\boldsymbol{A}(\boldsymbol{x}_k) + \boldsymbol{B}(\boldsymbol{x}_k)\boldsymbol{u}_k) - \boldsymbol{T}\boldsymbol{S}\boldsymbol{x}_k \\ &= (\boldsymbol{T} + \boldsymbol{P})\boldsymbol{S}\boldsymbol{A}(\boldsymbol{x}_k) + (\boldsymbol{T} + \boldsymbol{P})\boldsymbol{S}\boldsymbol{B}(\boldsymbol{x}_k)\boldsymbol{u}_k - \boldsymbol{T}\boldsymbol{S}\boldsymbol{x}_k = \boldsymbol{0} \end{aligned}$$

Hence, the optimal control law is

$$\boldsymbol{u}_k^* = -(\boldsymbol{S}\boldsymbol{B}(\boldsymbol{x}_k))^{-1}(\boldsymbol{T} + \boldsymbol{P})^{-1}(\boldsymbol{P}\boldsymbol{S}\boldsymbol{A}(\boldsymbol{x}_k) + \boldsymbol{T}\boldsymbol{S}(\boldsymbol{A}(\boldsymbol{x}_k) - \boldsymbol{x}_k))$$

We next use the Bellman equation (28) to obtain the matrix $\boldsymbol{P}$:

$$\boldsymbol{\sigma}_k^\top \boldsymbol{P}\boldsymbol{\sigma}_k = \boldsymbol{\sigma}_k^\top \boldsymbol{\sigma}_k + (\boldsymbol{\sigma}_{k+1} - \boldsymbol{\sigma}_k)^\top \boldsymbol{T}(\boldsymbol{\sigma}_{k+1} - \boldsymbol{\sigma}_k) + \boldsymbol{\sigma}_{k+1}^\top \boldsymbol{P}\boldsymbol{\sigma}_{k+1} \tag{30}$$

Collecting all quadratic terms in $\boldsymbol{\sigma}_k$ on the left side of Eq. (30), we obtain

$$\begin{aligned} \boldsymbol{\sigma}_k^\top(\boldsymbol{P} - \boldsymbol{I} - \boldsymbol{T})\boldsymbol{\sigma}_k &= \boldsymbol{\sigma}_{k+1}^\top \boldsymbol{T}\boldsymbol{\sigma}_{k+1} - 2\boldsymbol{\sigma}_{k+1}^\top \boldsymbol{T}\boldsymbol{\sigma}_k + \boldsymbol{\sigma}_{k+1}^\top \boldsymbol{P}\boldsymbol{\sigma}_{k+1} \\ &= \boldsymbol{\sigma}_{k+1}^\top(\boldsymbol{T} + \boldsymbol{P})\boldsymbol{\sigma}_{k+1} - 2\boldsymbol{\sigma}_{k+1}^\top \boldsymbol{T}\boldsymbol{\sigma}_k \end{aligned} \tag{31}$$

Because $(\boldsymbol{T} + \boldsymbol{P})\boldsymbol{\sigma}_{k+1} = \boldsymbol{T}\boldsymbol{\sigma}_k$, we can further simplify Eq. (31) to obtain

$$\begin{aligned} \boldsymbol{\sigma}_k^\top(\boldsymbol{P} - \boldsymbol{I} - \boldsymbol{T})\boldsymbol{\sigma}_k &= \boldsymbol{\sigma}_{k+1}^\top(\boldsymbol{T} + \boldsymbol{P})\boldsymbol{\sigma}_{k+1} - 2\boldsymbol{\sigma}_{k+1}^\top(\boldsymbol{T} + \boldsymbol{P})\boldsymbol{\sigma}_{k+1} \\ &= -\boldsymbol{\sigma}_{k+1}^\top(\boldsymbol{T} + \boldsymbol{P})\boldsymbol{\sigma}_{k+1} \\ &= -\boldsymbol{\sigma}_{k+1}^\top \boldsymbol{T}\boldsymbol{\sigma}_k \\ &= -\boldsymbol{\sigma}_k^\top \boldsymbol{T}^\top(\boldsymbol{T} + \boldsymbol{P})^{-1}\boldsymbol{T}\boldsymbol{\sigma}_k \end{aligned} \tag{32}$$

Rearranging terms in Eq. (32) gives the following matrix equation for $\boldsymbol{P}$:

$$\boldsymbol{P} = \boldsymbol{I} + \boldsymbol{T} - \boldsymbol{T}^\top(\boldsymbol{T} + \boldsymbol{P})^{-1}\boldsymbol{T} \tag{33}$$

□

Two difference equations [Eq. (14), derived from the calculus of variations, and Eq. (23), derived from the Bellman equation] are used to construct extremizing and minimizing control laws, respectively. It is not immediately obvious that these difference equations, along with their respective control laws (15) and (25), are equivalent. We now show that these difference equations are equivalent.

*Theorem 5:* The extremizing difference equation obtained from the calculus of variations is equivalent with the minimizing difference equation obtained from the Bellman equation because

$$\Lambda = \Lambda^{-} = (T + P)^{-1}T$$

so that

$$\sigma_{k+1} - (T + P)^{-1}T\sigma_k = \sigma_{k+1} - \Lambda\sigma_k = 0$$

that is, Eq. (23) takes the form of Eq. (14).

*Proof:* We add $T$ to both sides of Eq. (33) to obtain

$$T + P = I + 2T - T^{\top}(T + P)^{-1}T \tag{34}$$

Premultiplying both sides of Eq. (34) by $T^{-1}$ gives

$$T^{-1}(T + P) = T^{-1}(I + 2T) - (T + P)^{-1}T$$

Collecting all terms on one side yields

$$T^{-1}(T + P) - T^{-1}(I + 2T) + (T + P)^{-1}T = O \tag{35}$$

Let $\mathcal{Z}I = T^{-1}(T + P)$, then $\mathcal{Z}^{-1}I = (T + P)^{-1}T$, and Eq. (35) can be rewritten as

$$\mathcal{Z}I - T^{-1}(I + 2T) + \mathcal{Z}^{-1}I = O \tag{36}$$

Equation (36) is the same as Eq. (9) derived from the discrete Euler–Lagrange equation with the stabilizing solution given by $\mathcal{Z} = \Lambda^{-}$. Therefore, $\mathcal{Z}I = T^{-1}(T + P) = \Lambda^{-1}$. □

## V. Relationship Between Discrete-Time Synergetic Optimal Controller and the Linear Quadratic Regulator

The performance index used to derive the optimal control strategies (15) and (25) is quadratic. Optimal control of discrete-time linear time-invariant systems using a quadratic performance index has been extensively investigated in [8,9,34–36]. In this section, we establish connections between the discrete-time synergetic optimal control and the discrete-time LQR.

### A. Discrete-Time Synergetic Optimal Control for Linear Time-Invariant Systems

We first present the discrete-time synergetic control solution for a class of LTI systems.

*Corollary 1 (discrete-time synergetic optimal control for LTI systems):* Consider the linear discrete-time dynamic system modeled by

$$x_{k+1} = Ax_k + Bu_k, \qquad k_0 \le k < \infty, \qquad x_{k_0} = x_0 \tag{37}$$

and the associated performance index

$$J(x_0) = \sum_{k=k_0}^{\infty} \sigma_k^{\top}\sigma_k + (\sigma_{k+1} - \sigma_k)^{\top}T(\sigma_{k+1} - \sigma) \tag{38}$$

where $\sigma_k = Sx_k$, $x_k \in \mathbb{R}^n$, and $S$ is constructed such that $SB$ is invertible. The optimal control minimizing the performance index (38) satisfies the difference equation:

$$\sigma_{k+1} = (T + P)^{-1}T\sigma_k \tag{39}$$

and is given by

$$u_k^{*} = -((T + P)SB)^{-1}(PSA + TS(A - I))x_k \tag{40}$$

The symmetric positive-definite matrix $P \in \mathbb{R}^{m\times m}$ is the solution to the algebraic equation:

$$P = I + T - T^{\top}(T + P)^{-1}T \tag{41}$$

*Proof:* Substituting $A$ and $B$ for $A(x_k)$ and $B(x_k)$ into Theorem 4 gives the preceding results. □

To establish connections between the preceding synergetic optimal control law and the one obtained from the LQR theory, we need the following background results.

### B. Background Results from the LQR Theory

In this section, we investigate the optimal linear quadratic regulator problem with the cross-product term in the quadratic performance index. We derive a discrete LQR control law following the steps in [12], in which the solution to the continuous-time LQR problems with cross-product terms in the quadratic performance index can be found. Then we employ the obtained results to derive an optimal synergetic control strategy using the Bellman equation.

Consider a linear discrete-time dynamic system modeled by Eq. (37) and the associated quadratic performance index:

$$J(x_{k_0}) = \sum_{k=k_0}^{\infty} x_k^{\top}Qx_k + 2x_k^{\top}Nu_k + u_k^{\top}Ru_k \tag{42}$$

where $Q = Q^{\top} \ge 0$, $N \in \mathbb{R}^{n\times m}$, and $R = R^{\top} > 0$. The problem of determining an input sequence $u_k = u_k^{*}$ $(k \ge k_0)$ for the dynamic system (37) that minimizes $J(x_{k_0})$ is called the deterministic discrete-time linear quadratic optimal regulator (DT-LQR) problem.

*Lemma 1 (discrete-time LQR with cross-product term in quadratic performance index):* For the DT-LQR problem of dynamic system (37) with quadratic performance index $J(x_{k_0})$ given by Eq. (42), the control law

$$u_k^{*} = -(R + B^{\top}\tilde{P}B)^{-1}(N^{\top} + B^{\top}\tilde{P}A)x_k \tag{43}$$

minimizes the quadratic performance index $J(x_{k_0})$, where the positive-definite symmetric matrix $\tilde{P} \in \mathbb{R}^{n\times n}$ is the solution to the algebraic Riccati equation:

$$\tilde{P} = Q - (N + A^{\top}\tilde{P}B)(R + B^{\top}\tilde{P}B)^{-1}(N + A\tilde{P}B)^{\top} + A^{\top}\tilde{P}A \tag{44}$$

The control law $u_k^{*}$ given by Eq. (43) together with $\tilde{P}$ obtained from Eq. (44) solve the associated Bellman equation:

$$J^{*}(x_k) = \min_{u_k}\{x_k^{\top}Qx_k + 2x_k^{\top}Nu_k + u_k^{\top}Ru_k + J^{*}(x_{k+1})\} \tag{45}$$

for all $x_k \in \mathbb{R}^n$, where $J^{*}(x_k) = x_k^{\top}\tilde{P}x_k$.

*Proof:* Substituting $x_k^{\top}\tilde{P}x_k$ for $J(x_k)$ and

$$x_{k+1}^{\top}\tilde{P}x_{k+1} = (Ax_k + Bu_k)^{\top}\tilde{P}(Ax_k + Bu_k)$$

for $J(x_{k+1})$ in Eq. (45), then taking its derivative with respect to $u_k$ and solving the resulting algebraic equation for $u_k$, we obtain the optimizing control law (43). Substituting this optimizing control into Eq. (45) gives

$$x_k^{\top}\tilde{P}x_k = x_k^{\top}(Q - (N + A^{\top}\tilde{P}B)(R + B^{\top}\tilde{P}B)^{-1}(N + A\tilde{P}B)^{\top} + A^{\top}\tilde{P}A)x_k \tag{46}$$

For the equality in Eq. (46) to hold for all $x_k$, the algebraic Riccati equation (44) must be satisfied. □

### C. Equivalence Between Synergetic and LQR Controllers

We begin by substituting the expressions for $\sigma_k$ and $\sigma_{k+1}$ into the quadratic performance index (38) and then eliminate $x_{k+1}$. After some manipulations, we represent Eq. (38) in the form

$$J(x_{k_0}) = \sum_{k=k_0}^{\infty} x_k^{\top}Qx_k + 2x_k^{\top}Nu_k + u_k^{\top}Ru_k \tag{47}$$

where

$$\left.\begin{aligned} Q &= S^{\top}S + (A - I)^{\top}S^{\top}TS(A - I)\\ R &= B^{\top}S^{\top}TSB\\ N &= (A - I)^{\top}S^{\top}TSB \end{aligned}\right\} \tag{48}$$

By Lemma 1, the DT-LQR optimal solution for the system (37) with the cost functional (47) is given by Eq. (43), where $\tilde{P} \in \mathbb{R}^{n\times n}$ is the solution to the algebraic Riccati equation (44). We next show that the discrete-time synergetic optimal control law (40) and the discrete-time LQR control law (43) are equivalent optimal strategies associated with the quadratic performance index (38), expressed in terms of $\boldsymbol{\sigma}_k$, or with Eq. (47), expressed in terms of $\boldsymbol{x}_k$, respectively.

*Theorem 6:* The synergetic controller (40) and the DT-LQR controller (43) are equivalent: that is,

$$\begin{aligned}\boldsymbol{u}_k^* &= -(\boldsymbol{R}+\boldsymbol{B}^\top\tilde{\boldsymbol{P}}\boldsymbol{B})^{-1}(\boldsymbol{N}+\boldsymbol{A}^\top\tilde{\boldsymbol{P}}\boldsymbol{B})^\top\boldsymbol{x}_k\\ &= -((\boldsymbol{T}+\boldsymbol{P})\boldsymbol{S}\boldsymbol{B})^{-1}(\boldsymbol{P}\boldsymbol{S}\boldsymbol{A}+\boldsymbol{T}\boldsymbol{S}(\boldsymbol{A}-\boldsymbol{I}))\boldsymbol{x}_k\end{aligned}$$

where $\tilde{\boldsymbol{P}} = \boldsymbol{S}^\top\boldsymbol{P}\boldsymbol{S}$.

*Proof:* We first show that $\tilde{\boldsymbol{P}} = \boldsymbol{S}^\top\boldsymbol{P}\boldsymbol{S}$. The Bellman equation for the discrete-time synergetic optimal control of the LTI system model (37) with the performance index (38) is given by Eq. (38). Substituting into Eq. (28) the expressions for $\boldsymbol{\sigma}_k$ and $\boldsymbol{\sigma}_{k+1}$ and letting $J(\boldsymbol{x}_k) = \boldsymbol{x}_k^\top\boldsymbol{S}^\top\boldsymbol{P}\boldsymbol{S}\boldsymbol{x}_k$, we obtain

$$\begin{aligned}\boldsymbol{x}_k^\top\boldsymbol{S}^\top\boldsymbol{P}\boldsymbol{S}\boldsymbol{x}_k &= \boldsymbol{x}_k^\top\boldsymbol{S}^\top\boldsymbol{S}\boldsymbol{x}_k\\ &\quad+(\boldsymbol{A}\boldsymbol{x}_k+\boldsymbol{B}\boldsymbol{u}_k^*-\boldsymbol{x}_k)^\top\boldsymbol{S}^\top\boldsymbol{T}\boldsymbol{S}(\boldsymbol{A}\boldsymbol{x}_k+\boldsymbol{B}\boldsymbol{u}_k^*-\boldsymbol{x}_k)\\ &\quad+(\boldsymbol{A}\boldsymbol{x}_k+\boldsymbol{B}\boldsymbol{u}_k^*)^\top\boldsymbol{P}(\boldsymbol{A}\boldsymbol{x}_k+\boldsymbol{B}\boldsymbol{u}_k^*)\end{aligned}\tag{49}$$

We rearrange the terms to obtain

$$\begin{aligned}&\boldsymbol{x}_k^\top(\boldsymbol{S}^\top\boldsymbol{P}\boldsymbol{S}-\boldsymbol{A}^\top\boldsymbol{S}^\top\boldsymbol{P}\boldsymbol{S}\boldsymbol{A}-\boldsymbol{S}^\top\boldsymbol{S}-(\boldsymbol{A}-\boldsymbol{I})^\top\boldsymbol{S}^\top\boldsymbol{T}\boldsymbol{S}(\boldsymbol{A}-\boldsymbol{I}))\boldsymbol{x}_k\\ &\quad= 2\boldsymbol{x}_k^\top(\boldsymbol{A}^\top\boldsymbol{S}^\top\boldsymbol{P}\boldsymbol{S}\boldsymbol{B}+(\boldsymbol{A}-\boldsymbol{I})\boldsymbol{S}^\top\boldsymbol{T}\boldsymbol{S}\boldsymbol{B})\boldsymbol{u}_k^*\\ &\quad+\boldsymbol{u}_k^*(\boldsymbol{B}^\top\boldsymbol{S}^\top\boldsymbol{P}\boldsymbol{S}\boldsymbol{B}+\boldsymbol{B}^\top\boldsymbol{S}^\top\boldsymbol{T}\boldsymbol{S}\boldsymbol{B})\boldsymbol{u}_k^*\\ &\quad= 2\boldsymbol{x}_k^\top(\boldsymbol{A}^\top\boldsymbol{S}^\top\boldsymbol{P}+(\boldsymbol{A}-\boldsymbol{I})\boldsymbol{S}^\top\boldsymbol{T})\boldsymbol{S}\boldsymbol{B}\boldsymbol{u}_k^*\\ &\quad+\boldsymbol{u}_k^*\boldsymbol{B}^\top\boldsymbol{S}^\top(\boldsymbol{P}+\boldsymbol{T})\boldsymbol{S}\boldsymbol{B}\boldsymbol{u}_k^*\end{aligned}\tag{50}$$

Substituting into Eq. (50) the synergetic control law (40) gives

$$\begin{aligned}&\boldsymbol{S}^\top\boldsymbol{P}\boldsymbol{S}-\boldsymbol{A}^\top\boldsymbol{S}^\top\boldsymbol{P}\boldsymbol{S}\boldsymbol{A}-\boldsymbol{S}^\top\boldsymbol{S}-(\boldsymbol{A}-\boldsymbol{I})^\top\boldsymbol{S}^\top\boldsymbol{T}\boldsymbol{S}(\boldsymbol{A}-\boldsymbol{I})\\ &\quad=-(\boldsymbol{A}^\top\boldsymbol{S}^\top\boldsymbol{P}+(\boldsymbol{A}-\boldsymbol{I})\boldsymbol{S}^\top\boldsymbol{T})(\boldsymbol{T}+\boldsymbol{P})^{-1}(\boldsymbol{P}\boldsymbol{S}\boldsymbol{A}+\boldsymbol{T}\boldsymbol{S}(\boldsymbol{A}-\boldsymbol{I}))\\ &\quad=-(\boldsymbol{P}\boldsymbol{S}\boldsymbol{A}+\boldsymbol{T}\boldsymbol{S}(\boldsymbol{A}-\boldsymbol{I}))^\top(\boldsymbol{T}+\boldsymbol{P})^{-1}(\boldsymbol{P}\boldsymbol{S}\boldsymbol{A}+\boldsymbol{T}\boldsymbol{S}(\boldsymbol{A}-\boldsymbol{I}))\\ &\quad=-(\boldsymbol{P}\boldsymbol{S}\boldsymbol{A}+\boldsymbol{T}\boldsymbol{S}(\boldsymbol{A}-\boldsymbol{I}))^\top(\boldsymbol{S}\boldsymbol{B})(\boldsymbol{S}\boldsymbol{B})^{-1}(\boldsymbol{T}+\boldsymbol{P})^{-1}(\boldsymbol{S}\boldsymbol{B})^{-T}(\boldsymbol{S}\boldsymbol{B})^\top\\ &\quad\times(\boldsymbol{P}\boldsymbol{S}\boldsymbol{A}+\boldsymbol{T}\boldsymbol{S}(\boldsymbol{A}-\boldsymbol{I}))\\ &\quad=-(\boldsymbol{B}^\top\boldsymbol{S}^\top\boldsymbol{P}\boldsymbol{S}\boldsymbol{A}+\boldsymbol{B}^\top\boldsymbol{S}^\top\boldsymbol{T}\boldsymbol{S}(\boldsymbol{A}-\boldsymbol{I}))^\top(\boldsymbol{B}^\top\boldsymbol{S}^\top\boldsymbol{T}\boldsymbol{S}\boldsymbol{B}\\ &\quad+\boldsymbol{B}^\top\boldsymbol{S}^\top\boldsymbol{P}\boldsymbol{S}\boldsymbol{B})^{-1}\times(\boldsymbol{B}^\top\boldsymbol{S}^\top\boldsymbol{P}\boldsymbol{S}\boldsymbol{A}+\boldsymbol{B}^\top\boldsymbol{S}^\top\boldsymbol{T}\boldsymbol{S}(\boldsymbol{A}-\boldsymbol{I}))\end{aligned}\tag{51}$$

On the other hand, the solution to the DT-LQR problem of system (37) with the cost (47) is characterized by the matrix $\tilde{\boldsymbol{P}}$, which is the solution to the Riccati equation and has the form:

$$\tilde{\boldsymbol{P}}=\boldsymbol{Q}+\boldsymbol{A}^\top\tilde{\boldsymbol{P}}\boldsymbol{A}-(\boldsymbol{N}+\boldsymbol{A}^\top\boldsymbol{P}\boldsymbol{B})(\boldsymbol{R}+\boldsymbol{B}^\top\tilde{\boldsymbol{P}}\boldsymbol{B})^{-1}(\boldsymbol{N}+\boldsymbol{A}^\top\boldsymbol{P}\boldsymbol{B})^\top\tag{52}$$

We substitute the matrices $\boldsymbol{Q}$, $\boldsymbol{N}$, and $\boldsymbol{R}$ defined in Eq. (48) into Eq. (52) to obtain

$$\begin{aligned}\tilde{\boldsymbol{P}} &= \boldsymbol{S}^\top\boldsymbol{S}+(\boldsymbol{A}-\boldsymbol{I})^\top\boldsymbol{S}^\top\boldsymbol{T}\boldsymbol{S}(\boldsymbol{A}-\boldsymbol{I})+\boldsymbol{A}^\top\tilde{\boldsymbol{P}}\boldsymbol{A}\\ &\quad-((\boldsymbol{A}-\boldsymbol{I})^\top\boldsymbol{S}^\top\boldsymbol{T}\boldsymbol{S}\boldsymbol{B}+\boldsymbol{A}^\top\tilde{\boldsymbol{P}}\boldsymbol{B})(\boldsymbol{B}^\top\boldsymbol{S}^\top\boldsymbol{T}\boldsymbol{S}\boldsymbol{B}+\boldsymbol{B}^\top\tilde{\boldsymbol{P}}\boldsymbol{B})^{-1}\\ &\quad\times(\boldsymbol{B}^\top\boldsymbol{S}^\top\boldsymbol{T}\boldsymbol{S}(\boldsymbol{A}-\boldsymbol{I})+\boldsymbol{B}^\top\tilde{\boldsymbol{P}}\boldsymbol{A})\end{aligned}\tag{53}$$

Equations (51) and (53) are equal if and only if $\tilde{\boldsymbol{P}} = \boldsymbol{S}^\top\boldsymbol{P}\boldsymbol{S}$.

Next, we show that the control laws (40) and (43) are equivalent. Using matrix relations in Eq. (48) to substitute for $\boldsymbol{R}$ and $\boldsymbol{N}$ in the DT-LQR control law, we obtain

$$\begin{aligned}\boldsymbol{u}_k^* &= -(\boldsymbol{R}+\boldsymbol{B}^\top\tilde{\boldsymbol{P}}\boldsymbol{B})^{-1}(\boldsymbol{N}^\top+\boldsymbol{B}^\top\tilde{\boldsymbol{P}}\boldsymbol{A})\boldsymbol{x}_k\\ &= -(\boldsymbol{B}^\top\boldsymbol{S}^\top\boldsymbol{T}\boldsymbol{S}\boldsymbol{B}+\boldsymbol{B}^\top\tilde{\boldsymbol{P}}\boldsymbol{B})^{-1}(\boldsymbol{B}^\top\boldsymbol{S}^\top\boldsymbol{T}\boldsymbol{S}(\boldsymbol{A}-\boldsymbol{I})+\boldsymbol{B}^\top\tilde{\boldsymbol{P}}\boldsymbol{A})\boldsymbol{x}_k\end{aligned}\tag{54}$$

Substituting now $\boldsymbol{S}^\top\boldsymbol{P}\boldsymbol{S}$ for $\tilde{\boldsymbol{P}}$ into Eq. (54) and taking into account that $\boldsymbol{S}\boldsymbol{B}$ is invertible, the DT-LQR control law (54) takes the form of the synergetic control law (40). □

## VI. Discrete-Time Variable-Structure Sliding-Mode Control and Synergetic Control

The difference equation

$$\boldsymbol{\sigma}_{k+1} = \boldsymbol{\Lambda}\boldsymbol{\sigma}_k, \qquad 0 < \|\boldsymbol{\Lambda}\| < 1\tag{55}$$

derived in Secs. III and IV is an important equation in discrete-time synergetic control because it determines the associated discrete-time optimizing control law. Note that the relation (55) indicates that both $\boldsymbol{\sigma}_{k+1}$ and $\boldsymbol{\sigma}_k$ have the same sign. Recall that $\boldsymbol{\sigma}_k \in \mathbb{R}^m$. Let $\sigma_{i,k}$ be the $i$th element of $\boldsymbol{\sigma}_k$. The relation (55) is equivalent to

$$|\boldsymbol{\sigma}_{i,k+1}| < |\boldsymbol{\sigma}_{i,k}| \quad\text{or}\quad \begin{cases}(\boldsymbol{\sigma}_{i,k+1}-\boldsymbol{\sigma}_{i,k})\,\mathrm{sign}(\boldsymbol{\sigma}_{i,k})<0\\(\boldsymbol{\sigma}_{i,k+1}+\boldsymbol{\sigma}_{i,k})\,\mathrm{sign}(\boldsymbol{\sigma}_{i,k})>0,\end{cases}\tag{56}$$

where $\mathrm{sign}(\cdot)$ is the sign operator applied to each component of its vector argument. The relation (56) appears in discrete-time variable-structure sliding-mode control reaching conditions [37,38]. An overview of discrete-time sliding-mode control reaching conditions can be found in Chapter 3 of [39]. It is stated on page 931 of [38] that the control law satisfying Eq. (56), and hence Eq. (55), will guarantee that all the state trajectories will enter and remain within a domain of decreasing or, in the worst case, nonincreasing distance relative to the manifold $\mathcal{M} = \{\boldsymbol{x}:\boldsymbol{\sigma}(\boldsymbol{x}) = \boldsymbol{0}\}$.

The reaching law of the discrete-time sliding-mode control in the form of Eq. (55) is known as linear reaching law [39]. It is also known as the reaching law for variable structure with $\beta$-equivalent control [40]. The $\beta$-equivalent-control variable-structure method yields the equivalent control from the difference equation:

$$\boldsymbol{\sigma}_{k+1} = \beta\boldsymbol{\sigma}$$

where $|\beta| < 1$. From the perspective of discrete-time synergetic control, the discrete-time sliding-mode control with the linear or $\beta$ reaching law is optimizing, in the sense that it minimizes the cost functional:

$$J=\sum_{k=k_0}^{\infty}\boldsymbol{\sigma}_k^\top\boldsymbol{\sigma}_k+(\boldsymbol{\sigma}_{k+1}-\boldsymbol{\sigma}_k)^\top\boldsymbol{T}(\boldsymbol{\sigma}_{k+1}-\boldsymbol{\sigma}_k)$$

A consequence of recognizing a connection between the discrete-time synergetic control and the sliding-mode control is that we can employ invariant manifold construction methods of sliding-mode variable structure.

## VII. Analysis of the Closed-Loop System Driven by the Synergetic Controller

We consider the following LTI dynamic system model:

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}\boldsymbol{x}_k + \boldsymbol{B}\boldsymbol{u}_k$$

which is regulated by the *discrete-time synergetic optimal control*. The discrete-time synergetic optimal control law $\boldsymbol{u}_k$ is constructed to minimize the quadratic performance index (13), where $\boldsymbol{\sigma}_{k+1} = \boldsymbol{\Lambda}\boldsymbol{\sigma}_k$, which can be represented as

$$\boldsymbol{S}\boldsymbol{x}_{k+1} = \boldsymbol{S}(\boldsymbol{A}\boldsymbol{x}_k+\boldsymbol{B}\boldsymbol{u}_k) = \boldsymbol{\Lambda}\boldsymbol{S}\boldsymbol{x}_k\tag{57}$$

where $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}^-$ is given by Eq. (14); that is, all eigenvalues of $\boldsymbol{\Lambda}$ lie inside the unit disc in the complex plane. Therefore, the optimizing control law satisfying Eq. (57) is

$$\boldsymbol{u}_k = -(\boldsymbol{S}\boldsymbol{B})^{-1}(\boldsymbol{S}\boldsymbol{A}\boldsymbol{x}_k-\boldsymbol{\Lambda}\boldsymbol{S}\boldsymbol{x}_k)\tag{58}$$

The closed-loop system dynamics are modeled as

$$x_{k+1} = (I - B(SB)^{-1}S)Ax_k + B(SB)^{-1}\Lambda Sx_k \tag{59}$$

The synergetic control strategy (58) forces the system trajectory to asymptotically approach the manifold $\{\sigma = 0\}$. Thus, the closed-loop system will be asymptotically stable, provided that the system (59) restricted to the manifold, $\{\sigma = 0\}$, is asymptotically stable. In our proof of the preceding statements, we use the following theorem that can be found on page 333 of [41].

*Theorem 7:* Consider the system model $x_{k+1} = Ax_k + Bu_k$, where rank $B = m$, the pair $(A, B)$ is controllable, and a given set of $n - m$ complex numbers, $\{\lambda_1, \ldots, \lambda_{n-m}\}$, is symmetric with respect to the real axis. Then there exist full-rank matrices $W \in \mathbb{R}^{n\times(n-m)}$ and $W^g \in \mathbb{R}^{(n-m)\times n}$ such that $W^g W = I_{n-m}$ and $W^g B = O$, and the eigenvalues of $W^g AW$ are $\mathrm{eig}(W^g AW) = \{\lambda_1, \ldots, \lambda_{n-m}\}$. Furthermore, there exists a matrix $S \in \mathbb{R}^{m\times n}$ such that $SB = I_m$ and the system $x_{k+1} = Ax_k + Bu_k$ restricted to $\{Sx = 0\}$ has its poles at $\lambda_1, \ldots, \lambda_{n-m}$.

We can now state and prove the following theorem.

*Theorem 8:* If the manifold $\{Sx = 0\}$ is constructed so that the system $x_{k+1} = Ax_k + Bu_k$ restricted to it is asymptotically stable and $\det(SB) \neq 0$, then the closed-loop system

$$x_{k+1} = Ax_k + Bu_k, \qquad u_k = -(SB)^{-1}(SAx_k - \Lambda Sx_k)$$

is asymptotically stable.

*Proof:* Suppose that for a given set of complex numbers $\lambda_1, \ldots, \lambda_{n-m}$ for which the magnitudes are less than one, we constructed matrices $S \in \mathbb{R}^{m\times n}$, $W^g \in \mathbb{R}^{n\times(n-m)}$, and $W^g \in \mathbb{R}^{(n-m)\times n}$ that satisfy the conditions of Theorem 7. Let

$$z_k = \begin{bmatrix} z_{1,k} \\ z_{2,k} \end{bmatrix} = \begin{bmatrix} W^g \\ S \end{bmatrix} x_k \tag{60}$$

where $z_{1,k}$ is composed of state variables that characterize the closed-loop trajectories of Eq. (59) confined on the manifold $\sigma_k = 0$, and $z_{2,k}$ comprises state variables characterizing trajectories approaching the manifold $\sigma_k = 0$. The coordinate transformation matrix in Eq. (60) is invertible, where

$$[W \quad B]^{-1} = \begin{bmatrix} W^g \\ S \end{bmatrix} \tag{61}$$

and therefore $W^g W = I_{n-m}$, $SB = I_m$, $W^g B = O$, and $SW = O$. The closed-loop system dynamics expressed in the $z$ coordinates have the form

$$\begin{aligned} z_{k+1} &= \begin{bmatrix} W^g \\ S \end{bmatrix} x_{k+1} \\ &= \begin{bmatrix} W^g \\ S \end{bmatrix} [I - B(SB)^{-1}S]Ax_k + \begin{bmatrix} W^g \\ S \end{bmatrix} B(SB)^{-1}\Lambda Sx_k \\ &= \begin{bmatrix} W^g A - W^g B(SB)^{-1}SA \\ SA - SB(SB)^{-1}SA \end{bmatrix} x_k + \begin{bmatrix} W^g B(SB)^{-1}\Lambda S \\ SB(SB)^{-1}\Lambda S \end{bmatrix} x_k \end{aligned}$$

Because $SB = I_m$, $W^g B = O$, and $x_k = [W \quad B]z_k$, the preceding relation becomes

$$\begin{aligned} z_{k+1} &= \begin{bmatrix} W^g A \\ O \end{bmatrix} x_k + \begin{bmatrix} O \\ \Lambda S \end{bmatrix} x_k \\ &= \begin{bmatrix} W^g A \\ O \end{bmatrix} [W \quad B]z_k + \begin{bmatrix} O \\ \Lambda S \end{bmatrix} [W \quad B]z_k \\ &= \begin{bmatrix} W^g AW & W^g AB \\ O & O \end{bmatrix} z_k + \begin{bmatrix} O & O \\ O & \Lambda SB \end{bmatrix} z_k \\ &= \begin{bmatrix} W^g AW & W^g AB \\ O & \Lambda SB \end{bmatrix} z_k \end{aligned}$$

Therefore, the closed-loop dynamic system has the following final form:

$$z_{k+1} = \begin{bmatrix} W^g AW & W^g AB \\ O & \Lambda \end{bmatrix} z_k \tag{62}$$

The eigenvalues of the closed-loop system is the union of the eigenvalues of $W^g AW$ and $\Lambda$. Because the magnitudes of all eigenvalues of $\Lambda$ are less than one and because all eigenvalues of $W^g AW$ have magnitudes less than one, all eigenvalues of the closed-loop system have magnitudes less than one by construction, rendering asymptotic stability of the closed-loop system driven by the synergetic controller. □

The preceding analysis procedure can be extended to analyze the closed-loop stability of a class of nonlinear systems with matched nonlinearities (see, for example, page 548 in [42]). In particular, we consider a class of nonlinear systems that can be transformed into the so-called regular form:

$$\begin{aligned} \begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21}(x_k) & A_{22}(x_k) \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} + \begin{bmatrix} O \\ B_f \end{bmatrix} f(x_k) \\ &+ \begin{bmatrix} O \\ B_u \end{bmatrix} u_k = \tilde{A}(x_k)x_k + \tilde{B}_f f(x_k) + \tilde{B}_u u_k \end{aligned} \tag{63}$$

where $x_{1,k} \in \mathbb{R}^{n-m}$, $x_{2,k} \in \mathbb{R}^m$, $A_{21}(x_k) \in \mathbb{R}^{m\times(n-m)}$, and $A_{22}(x_k) \in \mathbb{R}^{m\times m}$ are state-dependent matrices; $f(x_k)$ is a $q$-dimensional vector of nonlinear functions; $B_f$ is a constant matrix of dimension $m \times q$; and $B_u \in \mathbb{R}^{m\times m}$ is a constant invertible matrix. Note that in system (63), nonlinearities are in the range space of the input matrix $B_u$; that is, they affect the system dynamics in the same way as the input $u_k$ does. Therefore, $B_f = B_u K_{fu}$, where $K_{fu}$ is a constant matrix of dimension $m \times q$.

Consider now the nonlinear system model (63) along with the performance index (3), where

$$\sigma_k = Sx_k = [S_1 \quad S_2] \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} \tag{64}$$

For the system (63), the control law that minimizes the performance index (3) is derived from the first-order difference equation (14) and is given by

$$u_k = -(S_2 B_u)^{-1}(S\tilde{A}(x_k)x_k + S_2 B_f f(x_k) - \Lambda Sx_k) \tag{65}$$

*Theorem 9 (closed-loop stability of dynamic system with matched nonlinearities):* If the matrix $S$ is constructed so that $\det(S_2 B_u) \neq 0$, system (63) restricted to the manifold $\{\sigma = 0\}$ is asymptotically stable, and the pair $(A_{11}, A_{12})$ is controllable, then the system (63) driven by the control law (65) is asymptotically stable.

*Proof:* We begin by constructing an $n \times (n - m)$ real matrix:

$$V = \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix}, \qquad V_{11} \neq O, \qquad V_{21} = O \tag{66}$$

where $V_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$ is invertible, such that $[V \ \tilde{B}_u]^{-1}$ exists. Let

$$[V \quad \tilde{B}_u]^{-1} = \begin{bmatrix} V_{11} & O \\ O & B_u \end{bmatrix}^{-1} = \begin{bmatrix} V_{11}^{-1} & O \\ O & B_u^{-1} \end{bmatrix} = \begin{bmatrix} V^g \\ \tilde{B}_u^g \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} V^g \\ \tilde{B}_u^g \end{bmatrix} [V \quad \tilde{B}_u] = \begin{bmatrix} V^g V & V^g \tilde{B}_u \\ \tilde{B}_u^g V & \tilde{B}_u^g \tilde{B}_u \end{bmatrix} = \begin{bmatrix} I_{n-m} & O \\ O & I_m \end{bmatrix} = I_n$$

and hence

$$V^g V = I_{n-m} \quad \text{and} \quad V^g \tilde{B}_u = O \tag{67}$$

Observe that the nonlinearities $A_{21}(x_k)$ and $A_{22}(x_k)$ are matched and, because of the structure of $V^g$, the matrix

$$V^g \tilde{A}(x_k) = [V_{11}^{-1} A_{11} \quad V_{11}^{-1} A_{12}]$$

is a constant matrix. It is easy to show that the pair $(\boldsymbol{A}_{11}, \boldsymbol{A}_{12})$ is controllable if and only if the pair

$$(\boldsymbol{V}^g \tilde{\boldsymbol{A}} \boldsymbol{V}, \boldsymbol{V}^g \tilde{\boldsymbol{A}} \tilde{\boldsymbol{B}}_u)$$

is controllable. Therefore, for any set of complex numbers $\{\lambda_1, \lambda_2, \ldots, \lambda_{n-m}\}$, symmetric with respect to the real axis, we can always find a matrix $\boldsymbol{F}$ such that

$$\{\lambda_1, \lambda_2, \ldots, \lambda_{n-m}\} = \text{eig}(\boldsymbol{V}^g \tilde{\boldsymbol{A}} \boldsymbol{V} – \boldsymbol{V}^g \tilde{\boldsymbol{A}} \tilde{\boldsymbol{B}}_u \boldsymbol{F})$$

The matrix $\boldsymbol{F}$ can be determined by using any available method for pole placing to shift the poles of $(\boldsymbol{V}^g \tilde{\boldsymbol{A}} \boldsymbol{V} – \boldsymbol{V}^g \tilde{\boldsymbol{A}} \tilde{\boldsymbol{B}}_u \boldsymbol{F})$ into the desired locations inside the open unit disc. We proceed by performing the following manipulations:

$$\boldsymbol{V}^g \tilde{\boldsymbol{A}} \boldsymbol{V} - \boldsymbol{V}^g \tilde{\boldsymbol{A}} \tilde{\boldsymbol{B}}_u \boldsymbol{F} = \boldsymbol{V}^g \tilde{\boldsymbol{A}} (\boldsymbol{V} - \tilde{\boldsymbol{B}}_u \boldsymbol{F}) \tag{68}$$

Let $\boldsymbol{W}^g = \boldsymbol{V}^g$ and $\boldsymbol{W} = \boldsymbol{V} - \tilde{\boldsymbol{B}}_u \boldsymbol{F}$. Note that by Eq. (67),

$$\boldsymbol{W}^g \boldsymbol{W} = \boldsymbol{V}^g (\boldsymbol{V} - \tilde{\boldsymbol{B}}_u \boldsymbol{F}) = \boldsymbol{V}^g \boldsymbol{V} = \boldsymbol{I}_{n-m} \tag{69}$$

We then use the same arguments as in the Proof of Theorem 8 and transform the closed-loop system dynamics into the new basis using the following state-space transformation:

$$\boldsymbol{z}_k = \begin{bmatrix} \boldsymbol{z}_{1,k} \\ \boldsymbol{z}_{2,k} \end{bmatrix} = \begin{bmatrix} \boldsymbol{W}^g \\ \boldsymbol{S} \end{bmatrix} \boldsymbol{x}_k \tag{70}$$

where $\boldsymbol{z}_{1,k}$ is composed of state variables that characterize the closed-loop trajectories of Eq. (63) confined on the manifold $\{\boldsymbol{\sigma} = \boldsymbol{0}\}$, and $\boldsymbol{z}_{2,k}$ comprises state variables characterizing trajectories approaching the manifold $\{\boldsymbol{\sigma} = \boldsymbol{0}\}$. The inverse transformation of Eq. (70) is

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{W}_g \\ \boldsymbol{S} \end{bmatrix}^{-1}, \qquad \boldsymbol{z}_k = [\boldsymbol{W} \quad \tilde{\boldsymbol{B}}_u] \boldsymbol{z}_k \tag{71}$$

The closed-loop system dynamics in the new coordinates are

$$\begin{aligned} \begin{bmatrix} \boldsymbol{z}_{1,k+1} \\ \boldsymbol{z}_{2,k+1} \end{bmatrix} &= \begin{bmatrix} \boldsymbol{W}^g \\ \boldsymbol{S} \end{bmatrix} (\tilde{\boldsymbol{A}}(\boldsymbol{x}_k) - \tilde{\boldsymbol{B}}_u (\boldsymbol{S}_2 \boldsymbol{B}_u)^{-1} (\boldsymbol{S} \tilde{\boldsymbol{A}}(\boldsymbol{x}_k) - \boldsymbol{\Lambda} \boldsymbol{S})) [\boldsymbol{W} \quad \tilde{\boldsymbol{B}}_u] \boldsymbol{z}_k \\ &+ \begin{bmatrix} \boldsymbol{W}^g \\ \boldsymbol{S} \end{bmatrix} (-\tilde{\boldsymbol{B}}_u (\boldsymbol{S}_2 \boldsymbol{B}_u(\boldsymbol{x}_k))^{-1} \boldsymbol{S}_2 \boldsymbol{B}_f + \tilde{\boldsymbol{B}}_f \boldsymbol{f}(\boldsymbol{x}_k)) \\ &= \begin{bmatrix} \boldsymbol{W}^g \tilde{\boldsymbol{A}}(\boldsymbol{x}_k) \boldsymbol{W} & \boldsymbol{W}^g \tilde{\boldsymbol{A}}(\boldsymbol{x}_k) \tilde{\boldsymbol{B}}_u \\ \boldsymbol{O} & \boldsymbol{\Lambda} \end{bmatrix} \boldsymbol{z}_k \end{aligned} \tag{72}$$

By construction, the eigenvalues of the constant matrix $\boldsymbol{W}^g \tilde{\boldsymbol{A}}(\boldsymbol{x}_k) \boldsymbol{W}$ are located inside the open unit disc. The eigenvalues of $\boldsymbol{\Lambda}$ are also located inside the open unit disc. Therefore, the closed-loop system (72) is asymptotically stable. □

Using the results of the preceding analysis, we can deduce an algorithm for constructing a linear manifold $\{\boldsymbol{S}\boldsymbol{x} = \boldsymbol{0}\}$ so that the nonlinear dynamics constrained to this manifold are asymptotically stable. This invariant manifold construction algorithm was first proposed in [43,44] for constructing a switching surface for sliding-mode controllers:

1) Select $\boldsymbol{V}$ so that the rank of $[\boldsymbol{V} \; \boldsymbol{B}]$ is $n$.
2) Determine the matrix $\boldsymbol{F}$ so that the eigenvalues of $\boldsymbol{V}^g \boldsymbol{A} \boldsymbol{V} - \boldsymbol{V}^g \boldsymbol{A} \boldsymbol{B} \boldsymbol{F}$ are at desirable locations.
3) Set $\boldsymbol{W} = \boldsymbol{V} - \boldsymbol{B}\boldsymbol{F}$.
4) Determine the matrix $\boldsymbol{S}$ as the last $m$ rows of the inverse of $[\boldsymbol{W} \; \boldsymbol{B}]$.

## VIII. Discrete-Time Synergetic Optimal Integral Tracking Control

In this section, we use the results from the previous sections to develop a discrete-time integral tracking controller for a class of nonlinear systems that can be transformed into a regular form [Eq. (63)]. Suppose that the output of the system (63) is

$$\boldsymbol{y}_k = [\boldsymbol{C}_1 \quad \boldsymbol{C}_2] \begin{bmatrix} \boldsymbol{x}_{1,k} \\ \boldsymbol{x}_{2,k} \end{bmatrix} \tag{73}$$

where $\boldsymbol{y}_k \in \mathbb{R}^p$. The control design objective is to construct a control strategy so that the output $\boldsymbol{y}_k$ tracks a piecewise constant reference vector $\boldsymbol{r} \in \mathbb{R}^p$. We now define a new state vector $\boldsymbol{x}_{r,k}$ and its respective state equation:

$$\boldsymbol{x}_{r,k+1} = \boldsymbol{x}_{r,k} + \Gamma(\boldsymbol{r}_k - \boldsymbol{y}_k) \tag{74}$$

where $\Gamma > 0$ is a design parameter. Our objective is to design a control strategy so that

$$\lim_{k\to\infty} \boldsymbol{x}_{r,k+1} = \boldsymbol{x}_{r,k}$$

Therefore, $\boldsymbol{r}_\infty = \boldsymbol{y}_\infty$, and the performance index (3) is minimized, where $\boldsymbol{\sigma}_k$ is to be determined.

To proceed, define the augmented state-variable vector as

$$\tilde{\boldsymbol{x}}_k = \begin{bmatrix} \boldsymbol{x}_{r,k} \\ \boldsymbol{x}_{1,k} \\ \boldsymbol{x}_{2,k} \end{bmatrix} \in \mathbb{R}^N$$

where $N = n + p$. The augmented system dynamics are governed by the state equation:

$$\begin{bmatrix} \boldsymbol{x}_{r,k+1} \\ \boldsymbol{x}_{1,k+1} \\ \boldsymbol{x}_{2,k+2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & -\Gamma \boldsymbol{C}_1 & -\Gamma \boldsymbol{C}_2 \\ \boldsymbol{O} & \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{O} & \boldsymbol{A}_{21}(\boldsymbol{x}_k) & \boldsymbol{A}_{22}(\boldsymbol{x}_k) \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_{r,k} \\ \boldsymbol{x}_{1,k} \\ \boldsymbol{x}_{2,k} \end{bmatrix} + \begin{bmatrix} \Gamma \boldsymbol{I} \\ \boldsymbol{O} \\ \boldsymbol{O} \end{bmatrix} \boldsymbol{r}_k + \begin{bmatrix} \boldsymbol{O} \\ \boldsymbol{O} \\ \boldsymbol{B}_f \end{bmatrix} \boldsymbol{f}(\boldsymbol{x}_k) + \begin{bmatrix} \boldsymbol{O} \\ \boldsymbol{O} \\ \boldsymbol{B}_u \end{bmatrix} \boldsymbol{u}_k$$

that can be compactly expressed as

$$\tilde{\boldsymbol{x}}_{k+1} = \tilde{\boldsymbol{A}}(\boldsymbol{x}_k) \tilde{\boldsymbol{x}}_k + \tilde{\boldsymbol{B}}_r \boldsymbol{r}_k + \tilde{\boldsymbol{B}}_f \boldsymbol{f}(\boldsymbol{x}_k) + \tilde{\boldsymbol{B}}_u \boldsymbol{u}_k \tag{75}$$

Note that for the system (75), the matching condition implies that there exists a constant matrix $\boldsymbol{K}_{fu} \in \mathbb{R}^{m\times q}$ such that $\boldsymbol{B}_f = \boldsymbol{B}_u \boldsymbol{K}_{fu}$ as was demonstrated for Eq. (63). Therefore, $\tilde{\boldsymbol{B}}_f = \tilde{\boldsymbol{B}}_u \boldsymbol{K}_{fu}$. Let $\boldsymbol{\sigma}_k$ be defined as

$$\boldsymbol{\sigma}_k = [\boldsymbol{S}_r \quad \boldsymbol{S}_1 \quad \boldsymbol{S}_2] \tilde{\boldsymbol{x}}_k = \boldsymbol{S} \tilde{\boldsymbol{x}}_k \tag{76}$$

We next employ the equations $\boldsymbol{\sigma}_{k+1} = \boldsymbol{\Lambda} \boldsymbol{\sigma}_k$ and $\boldsymbol{\sigma}_k = \boldsymbol{S} \tilde{\boldsymbol{x}}_k$ to construct the discrete-time synergetic optimal control law. Combining the preceding two equations and rearranging the resulting equation gives

$$\begin{aligned} \boldsymbol{0} &= \boldsymbol{S} \tilde{\boldsymbol{x}}_{k+1} - \boldsymbol{\Lambda} \boldsymbol{\sigma}_k \\ &= \boldsymbol{S}(\tilde{\boldsymbol{A}}(\boldsymbol{x}_k) \tilde{\boldsymbol{x}}_k + \tilde{\boldsymbol{B}}_r \boldsymbol{r}_k + \tilde{\boldsymbol{B}}_f \boldsymbol{f}(\boldsymbol{x}_k) + \tilde{\boldsymbol{B}}_u \boldsymbol{u}_k) - \boldsymbol{\Lambda} \boldsymbol{\sigma}_k \end{aligned}$$

Solving the preceding equation for $\boldsymbol{u}_k$, we obtain

$$\boldsymbol{u}_k = -(\boldsymbol{S} \tilde{\boldsymbol{B}}_u)^{-1} (\boldsymbol{S} \tilde{\boldsymbol{A}}(\boldsymbol{x}_k) \tilde{\boldsymbol{x}}_k + \boldsymbol{S} \tilde{\boldsymbol{B}}_r \boldsymbol{r}_k + \boldsymbol{S} \tilde{\boldsymbol{B}}_f \boldsymbol{f}(\boldsymbol{x}_k) - \boldsymbol{\Lambda} \boldsymbol{\sigma}_k) \tag{77}$$

Substituting the control law (77) into Eq. (75) leads to the following closed-loop system model:

$$\begin{aligned} \tilde{\boldsymbol{x}}_{k+1} &= (\boldsymbol{I} - \tilde{\boldsymbol{B}}_u (\boldsymbol{S} \tilde{\boldsymbol{B}}_u)^{-1} \boldsymbol{S}) \tilde{\boldsymbol{A}}(\boldsymbol{x}_k) + \tilde{\boldsymbol{B}}_u (\boldsymbol{S} \tilde{\boldsymbol{B}}_u)^{-1} \boldsymbol{\Lambda} \boldsymbol{S} \tilde{\boldsymbol{x}}_k \\ &\quad - \tilde{\boldsymbol{B}}_u (\boldsymbol{S} \tilde{\boldsymbol{B}}_u)^{-1} \boldsymbol{S} \tilde{\boldsymbol{B}}_r \boldsymbol{r}_k + \tilde{\boldsymbol{B}}_r \boldsymbol{r}_k \end{aligned} \tag{78}$$

We now use the method advanced in the previous section to analyze the closed-loop tracking-system stability. In particular, we employ the same coordinate transformation (60) that maps $\boldsymbol{x}_k$ into the $[\boldsymbol{z}_{1,k}^\top \ \boldsymbol{z}_{2,k}^\top]^\top$-coordinate system. However, matrix dimensions are adjusted so that $\boldsymbol{W}^g \in \mathbb{R}^{(N-m)\times N}$ and $\boldsymbol{S} \in \mathbb{R}^{m\times N}$. The inverse transformation that maps $\boldsymbol{z}_k$ into $\tilde{\boldsymbol{x}}_k$ is given by Eq. (61), where matrix dimensions for the tracking problems are adjusted so that $\boldsymbol{W} \in \mathbb{R}^{N\times(N-m)}$ and $\tilde{\boldsymbol{B}}_u \in \mathbb{R}^{N\times m}$. Applying the preceding transformation to Eq. (78) yields

$$\begin{aligned}\begin{bmatrix} z_{1,k+1} \\ z_{2,k+1} \end{bmatrix} &= \begin{bmatrix} \boldsymbol{W}^g\tilde{\boldsymbol{A}}(\boldsymbol{x}_k)\boldsymbol{W} & \boldsymbol{W}^g\tilde{\boldsymbol{A}}(\boldsymbol{x}_k)\tilde{\boldsymbol{B}}_u \\ \boldsymbol{O} & \boldsymbol{\Lambda} \end{bmatrix}\begin{bmatrix} z_{1,k} \\ z_{2,k} \end{bmatrix} + \begin{bmatrix} \boldsymbol{W}^g \\ \boldsymbol{S} \end{bmatrix}(-\tilde{\boldsymbol{B}}_u(\boldsymbol{S}\tilde{\boldsymbol{B}}_u)^{-1}\boldsymbol{S}\tilde{\boldsymbol{B}}_r + \tilde{\boldsymbol{B}}_r)\boldsymbol{r}_k \\ &= \begin{bmatrix} \boldsymbol{W}^g\tilde{\boldsymbol{A}}(\boldsymbol{x}_k)\boldsymbol{W} & \boldsymbol{W}^g\tilde{\boldsymbol{A}}(\boldsymbol{x}_k)\tilde{\boldsymbol{B}}_u \\ \boldsymbol{O} & \boldsymbol{\Lambda} \end{bmatrix}\begin{bmatrix} z_{1,k} \\ z_{2,k} \end{bmatrix} + \begin{bmatrix} \boldsymbol{O} + \boldsymbol{W}^g\tilde{\boldsymbol{B}}_r \\ -\boldsymbol{S}\tilde{\boldsymbol{B}}_r + \boldsymbol{S}\tilde{\boldsymbol{B}}_r \end{bmatrix}\boldsymbol{r}_k \\ &= \begin{bmatrix} \boldsymbol{W}^g\tilde{\boldsymbol{A}}(\boldsymbol{x}_k)\boldsymbol{W} & \boldsymbol{W}^g\tilde{\boldsymbol{A}}(\boldsymbol{x}_k)\tilde{\boldsymbol{B}}_u \\ \boldsymbol{O} & \boldsymbol{\Lambda} \end{bmatrix}\begin{bmatrix} z_{1,k} \\ z_{2,k} \end{bmatrix} + \begin{bmatrix} \boldsymbol{W}^g\tilde{\boldsymbol{B}}_r \\ \boldsymbol{O} \end{bmatrix}\boldsymbol{r}_k \end{aligned} \tag{79}$$

Because the plant model is in the regular form with matched nonlinearities, the matrix $\boldsymbol{W}^g\tilde{\boldsymbol{A}}(\boldsymbol{x}_k)\boldsymbol{W}$ is a constant matrix. Observe that in the absence of $\boldsymbol{r}_k$, the dynamics of Eq. (79) are asymptotically stable because the magnitudes of all eigenvalues of $\boldsymbol{\Lambda}$ are less than one and, by construction, the magnitudes of all eigenvalues of the constant matrix $\boldsymbol{W}^g\tilde{\boldsymbol{A}}(\boldsymbol{x}_k)\boldsymbol{W}$ are also less than one. In the steady state, the following conditions hold:

$$\boldsymbol{z}_{2,k} = \boldsymbol{0} \tag{80}$$

$$\boldsymbol{z}_{1,k} = \boldsymbol{W}^g\tilde{\boldsymbol{A}}(\boldsymbol{x}_k)\boldsymbol{W}\boldsymbol{z}_{1,k} + \boldsymbol{W}^g\tilde{\boldsymbol{B}}_r\boldsymbol{r}_k \tag{81}$$

We will now show that the condition (81) is equivalent to the condition

$$\boldsymbol{y}_k = \boldsymbol{r}_k \tag{82}$$

We proceed as follows. Without loss of generality, let the matrices $\boldsymbol{V}_{11}$ in Eq. (66) and $\boldsymbol{F}$ in Eqs. (68) and (69) be defined as

$$\boldsymbol{V}_{11} = \begin{bmatrix} \boldsymbol{I}_p & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{I}_{n-m} \end{bmatrix} \quad \text{and} \quad \boldsymbol{F} = [\boldsymbol{F}_1 \quad \boldsymbol{F}_2] \tag{83}$$

The matrices $\boldsymbol{V}_{11}$ and $\boldsymbol{F}$ defined in Eq. (83) yield the following matrices $\boldsymbol{W}$ and $\boldsymbol{W}^g$:

$$\boldsymbol{W} = \begin{bmatrix} \boldsymbol{I}_p & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{I}_{n-m} \\ -\boldsymbol{B}_u\boldsymbol{F}_1 & -\boldsymbol{B}_u\boldsymbol{F}_2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{W}^g = \begin{bmatrix} \boldsymbol{I}_p & \boldsymbol{O}_{p\times(n-m)} & \boldsymbol{O}_{p\times m} \\ \boldsymbol{O}_{(n-m)\times p} & \boldsymbol{I}_{n-m} & \boldsymbol{O}_{(n-m)\times m} \end{bmatrix} \tag{84}$$

From the matrix identity

$$[\boldsymbol{W} \quad \tilde{\boldsymbol{B}}_u]\begin{bmatrix} \boldsymbol{W}^g \\ \boldsymbol{S} \end{bmatrix} = \boldsymbol{I}_N$$

we obtain the following relations:

$$\boldsymbol{B}_u\boldsymbol{F}_1 = \boldsymbol{B}_u\boldsymbol{S}_r \quad \text{and} \quad \boldsymbol{B}_u\boldsymbol{F}_2 = \boldsymbol{B}_u\boldsymbol{S}_1 \tag{85}$$

Using the matrices defined in Eq. (84) and taking into account the relation that follows from Eq. (70), which has the form $\boldsymbol{z}_1 = \boldsymbol{W}^g[\boldsymbol{x}_r^\top \quad \boldsymbol{x}_1^\top]^\top$, we can represent Eq. (81) as

$$\begin{bmatrix} \boldsymbol{x}_r \\ \boldsymbol{x}_1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_p + \Gamma\boldsymbol{C}_2\boldsymbol{B}_u\boldsymbol{F}_1 & -\Gamma\boldsymbol{C}_1 + \Gamma\boldsymbol{C}_2\boldsymbol{B}_u\boldsymbol{F}_2 \\ -\boldsymbol{A}_{12}\boldsymbol{B}_u\boldsymbol{F}_1 & \boldsymbol{A}_{11} - \boldsymbol{A}_{12}\boldsymbol{B}_u\boldsymbol{F}_2 \end{bmatrix}\begin{bmatrix} \boldsymbol{x}_r \\ \boldsymbol{x}_1 \end{bmatrix} + \begin{bmatrix} \Gamma\boldsymbol{I}_p \\ \boldsymbol{O} \end{bmatrix}\boldsymbol{r} \tag{86}$$

where we removed the subscript $k$ to indicate that we analyze the steady-state condition. In the steady state, $\boldsymbol{\sigma}(\tilde{\boldsymbol{x}}_k) = \boldsymbol{0}$; therefore,

$$\boldsymbol{S}_r\boldsymbol{x}_r + \boldsymbol{S}_1\boldsymbol{x}_1 + \boldsymbol{S}_2\boldsymbol{x}_2 = \boldsymbol{0} \tag{87}$$

From the first row of Eq. (86), we obtain

$$\boldsymbol{C}_2\boldsymbol{B}_u\boldsymbol{F}_1\boldsymbol{x}_r - (\boldsymbol{C}_1 - \boldsymbol{C}_2\boldsymbol{B}_u\boldsymbol{F}_2)\boldsymbol{x}_1 + \boldsymbol{r} = \boldsymbol{0} \tag{88}$$

Substituting Eqs. (85) and (87) into Eq. (88) and performing some algebraic manipulations gives

$$\begin{aligned} &\boldsymbol{C}_2\boldsymbol{B}_u\boldsymbol{S}_r\boldsymbol{x}_r - \boldsymbol{C}_1\boldsymbol{x}_1 + \boldsymbol{C}_2\boldsymbol{B}_u\boldsymbol{S}_1\boldsymbol{x}_1 + \boldsymbol{r} \\ &\quad = \boldsymbol{C}_2\boldsymbol{B}_u(-\boldsymbol{S}_1\boldsymbol{x}_1 - \boldsymbol{S}_2\boldsymbol{x}_2) - \boldsymbol{C}_1\boldsymbol{x}_1 + \boldsymbol{C}_2\boldsymbol{B}_u\boldsymbol{S}_1\boldsymbol{x}_1 + \boldsymbol{r} \\ &\quad = -\boldsymbol{C}_2\boldsymbol{x}_2 - \boldsymbol{C}_1\boldsymbol{x}_1 + \boldsymbol{r} \\ &\quad = -\boldsymbol{y} + \boldsymbol{r} \\ &\quad = \boldsymbol{0} \end{aligned}$$

Hence, in the steady state, $\boldsymbol{y}_k = \boldsymbol{r}_k$.

We summarize the results of this section in the following theorem.

*Theorem 10 (discrete-time synergetic optimal integral tracking):* If the following conditions are satisfied, then the system (75) driven by the control strategy (77) asymptotically tracks a constant reference signal $\boldsymbol{r}_k$:

1) The matrix $\boldsymbol{S}$ in Eq. (76) is constructed so that $\det(\boldsymbol{S}_2\boldsymbol{B}_u) \neq 0$.
2) The system (75) restricted to the manifold $\{\tilde{\boldsymbol{x}}_k:\boldsymbol{\sigma} = \boldsymbol{0}\}$ is asymptotically stable.
3) The pair $(\boldsymbol{A}_{11}, \boldsymbol{A}_{12})$ is controllable.

## IX. Example

We consider the problem of tracking control design for a nonlinear helicopter model using the proposed discrete-time synergetic control law. We employ the continuous-time nonlinear helicopter model developed by Pallet et al. [45–47]. In Fig. 1, we present the configuration of this helicopter control model.

### A. Nonlinear Dynamics

The vertical dynamics of the helicopter system mounted on a stand are governed by the following set of differential equations:

$$\ddot{h} = \omega^2[a_1 + a_2\theta_c - \sqrt{a_3 + a_4\theta_c}] + a_5\dot{h} + a_6\dot{h}^2 + a_7 \tag{89}$$

$$\dot{\omega} = a_8\omega + a_{10}\omega^2\sin\theta_c + a_9\omega^2 + a_{11} + u_1 \tag{90}$$

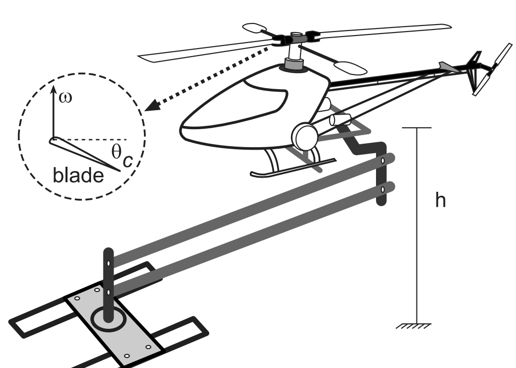


**Fig. 1 Helicopter system on a stand as modeled in [45–47].**

$$\dot{\theta}_c = a_{13}\theta_c + a_{14}\omega^2 \sin\theta_c + a_{15}\dot{\theta}_c + a_{12} + u_2 \tag{91}$$

where $h$ denotes the helicopter altitude in meters, $\omega$ denotes the rotor blade angular velocity in rad/s, and $\theta_c$ denotes the collective pitch angle of rotor blades in radians. The helicopter plant uses two actuators, rotary wing throttle $u_1$ and collective wing blade pitch $u_2$. The parameters for the nonlinear model [Eqs. (89–91)] are determined in [45–47] and are given in Table 1. The system outputs for this problem are the helicopter altitude $h$ and the collective pitch angle of the rotor blades $\theta_c$. To achieve a set of smooth control command actuations, integrators are inserted in front of the two input channels as in [48,49], so that

$$\dot{u}_1 = v_1 \quad \text{and} \quad \dot{u}_2 = v_2 \tag{92}$$

respectively. The variables $v_1$ and $v_2$ also serve as auxiliary control variables in the following control design process. Adding integrators to the dynamic equations (89–91) increases the overall order of helicopter system dynamics. This operation of adding integrators yields the resulting system dynamics in the so-called extended form [50,51].

### B. Development of a Discrete-Time State-Space Helicopter Dynamic Model

Define the following state vector $z$ such that

$$\left.\begin{array}{c} z_1 = h \\ z_2 = \dot{h} \\ z_3 = \omega^2[a_1 + a_2\theta_c - \sqrt{a_3 + a_4\theta_c}] + a_5\dot{h} + a_6\dot{h}^2 + a_7 \\ z_4 = 2\omega[a_1 + a_2\theta_c - \sqrt{a_3 + a_4\theta_c}](a_8\omega + a_{10}\omega^2\sin\theta_c + a_9\omega^2 + a_{11} + u_1) \\ +(a_5 + 2a_6\dot{h})(\omega^2[a_1 + a_2\theta_c - \sqrt{a_3 + a_4\theta_c}] + a_5\dot{h} + a_6\dot{h}^2 + a_7) \\ +\omega^2[a_2\dot{\theta}_c - \frac{1}{2}a_4\dot{\theta}_c(a_3 + a_4\theta_c)^{-1/2}] \\ z_5 = \theta_c \\ z_6 = \dot{\theta}_c \\ z_7 = a_{13}\theta_c + a_{14}\omega^2\sin\theta_c + a_{15}\dot{\theta}_c + a_{12} \end{array}\right\} \tag{93}$$

In terms of elements of the state vector $z$, the physical variables of the helicopter dynamics are

$$\left.\begin{array}{c} h = z_1 \\ \dot{h} = z_2 \\ \omega = \zeta_1 \\ \theta_c = z_5 \\ \dot{\theta}_c = z_6 \\ u_1 = \frac{z_4 - (a_5 + 2a_6z_2)[\zeta_1^2(a_1 + a_2 + z_5 - \sqrt{a_3 + a_4z_5}) + a_5z_2 + a_6z_2^2 + a_7]}{2\zeta_1(a_1 + a_2 + z_5 - \sqrt{a_3 + a_4z_5})} - \frac{\zeta_1^2(a_2z_6 - \frac{1}{2}z_6(a_3 + a_4z_5)^{-1/2})}{2\zeta_1(a_1 + a_2 + z_5 - \sqrt{a_3 + a_4z_5})} - a_8\zeta_1 - a_{10}\zeta_1^2\sin z_5 - a_9\zeta_1^2 - a_{11} \\ u_2 = z_7 - a_{13}z_5 - a_{14}\frac{z_3 - a_5z_2 - a_6z_2^2 - a_7}{a_1 + a_2z_5 - \sqrt{a_3 + a_4z_5}}\sin z_5 - a_{15}z_6 - a_{12} \end{array}\right\} \tag{94}$$

where

$$\zeta_1 = \sqrt{\frac{z_3 - a_5z_2 - a_6z_2^2 - a_7}{a_1 + a_2z_5 - \sqrt{a_3 + a_4z_5}}}$$

To proceed with the state-space description of the helicopter dynamics, we define

$$\zeta_2 = \frac{z_4 - (a_5 + 2a_6z_2)z_3 - \zeta_1^2[a_2z_6 - \frac{1}{2}a_4z_6(a_3 + a_4z_5)^{-1/2}]}{2\zeta_1[a_1 + a_2z_5 - \sqrt{a_3 + a_4z_5}]}$$

The helicopter vertical motion dynamics in the $z$ coordinates have the following state-space form:

$$\left.\begin{array}{c} \dot{z}_1 = z_2 \\ \dot{z}_2 = z_3 \\ \dot{z}_3 = z_4 \\ \dot{z}_4 = A_4^z(z) + B_4^z(z)v_1 \\ \dot{z}_5 = z_6 \\ \dot{z}_6 = z_6 \\ \dot{z}_7 = A_7^z(z) + B_7^z(z)v_2 \end{array}\right\} \tag{95}$$

where

$$\begin{aligned} A_4^z(z) &= [2\zeta_2^2 + 2\zeta_1(a_8\zeta_2 + 2a_{10}\zeta_1\zeta_2\sin z_5 + a_{10}\zeta_1^2z_6\cos z_5 \\ &\quad + 2a_9\zeta_1\zeta_2)] \times [a_1 + a_2z_5 - \sqrt{a_3 + a_4z_5}] + 4\zeta_1\zeta_2[a_2z_6 \\ &\quad - \tfrac{1}{2}a_4z_6(a_3 + a_4z_5)^{-1/2}] + 2a_6z_3^2 + (a_5 + 2a_6z_2)z_5 + \zeta_1^2[a_2z_7 \\ &\quad - \tfrac{1}{2}a_4z_7(a_3 + a_4z_2)^{-1/2} + \tfrac{1}{4}a_4^2z_6^2(a_3 + a_4z_5)]^{-1/2}, \\ B_4^z(z) &= 2\zeta_1[a_1 + a_2z_5 - \sqrt{a_3 + a_4z_5}], \\ A_7^z(z) &= a_{13}z_6 + 2a_{14}\zeta_1\zeta_2\sin z_5 + a_{14}\zeta_1^2z_6\cos z_5 + a_{15}z_7, \\ B_7^z(z) &= 1 \end{aligned}$$

The output equation for the state-space model (95) is

$$y = \begin{bmatrix} h \\ \theta_c \end{bmatrix} = \begin{bmatrix} z_1 \\ z_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} z = Cz \tag{96}$$

The control design for the helicopter vertical dynamics employs an integral tracking approach to achieve zero steady-state tracking errors. For each of the output variables in Eq. (96), we introduce additional state variables representing the integral of tracking error. Consequently, the helicopter vertical dynamics design model (95) is augmented by the following state equations:

**Table 1 Parameters for helicopter vertical dynamics**

| | | |
|---|---|---|
| $a_1 = 5.31 \times 10^4$ | $a_6 = 0.1$, s$^{-1}$ | $a_{11} = -13.92$, s$^{-1}$ |
| $a_2 = 1.5364 \times 10^{-2}$ | $a_7 = -g - 7.86$, m/s$^2$ | $a_{12} = 0.5436 \times 800.0$, s$^{-2}$ |
| $a_3 = 2.82 \times 10^{-7}$ | $a_8 = -0.7$, s$^{-1}$ | $a_{13} = -800.0$, s$^{-2}$ |
| $a_4 = 1.632 \times 10^{-5}$ | $a_9 = -0.0028$ | $a_{14} = -0.1$ |
| $a_5 = -0.1$, s$^{-1}$ | $a_{10} = -0.0028$ | $a_{15} = -65.0$, s$^{-1}$ |

$$\dot{z}_{h_r} = z_1 - h_r, \qquad \dot{z}_{\theta_{c,r}} = z_5 - \theta_{c,r} \tag{97}$$

We next employ the Euler discretization method to obtain a discrete-time model of the nonlinear augmented system (95–97). The time-continuous model is nonlinear and therefore, unlike in the linear case, we do not have available to us discretization formulas that yield an exact discrete-time model. We observe, however, that the system dynamics (95) and (97) are locally Lipschitz, which makes the discrete-time Euler approximate model close to the exact discrete-time model of the dynamics (95) and (97). For more details on the subject of discretization of nonlinear models, we refer to [52,53].

We now define the following discrete-time state variables and their corresponding continuous-time state variables:

$$\begin{bmatrix} x_{1,k} & x_{2,k} & x_{3,k} & x_{4,k} & x_{5,k} & x_{6,k} & x_{7,k} & x_{8,k} & x_{9,k} \end{bmatrix}^\top = \begin{bmatrix} z_{h_r}(k\tau) & z_1(k\tau) & z_2(k\tau) & z_3(k\tau) & z_4(k\tau) & z_{\theta_{c,r}}(k\tau) & z_5(k\tau) & z_6(k\tau) & z_7(k\tau) \end{bmatrix}^\top$$

where $\tau$ denotes the sampling time and $k$ denotes the sampling sequence index. Let $h_{r,k}$ and $\theta_{c,r,k}$ be the respective values of $h_r$ and $\theta_{c,r}$ at time $t = \tau k$. Using the previously defined state variables and references, the discrete-time Euler approximation model for the helicopter vertical dynamics is

$$\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \\ x_{3,k+1} \\ x_{4,k+1} \\ x_{5,k+1} \\ x_{6,k+1} \\ x_{7,k+1} \\ x_{8,k+1} \\ x_{9,k+1} \end{bmatrix} = \begin{bmatrix} x_{1,k} + \tau x_{2,k} \\ x_{2,k} + \tau x_{3,k} \\ x_{3,k} + \tau x_{4,k} \\ x_{4,k} + \tau x_{5,k} \\ x_{5,k} + \tau A_5^x(\boldsymbol{x}_k) \\ x_{6,k} + \tau x_{7,k} \\ x_{7,k} + \tau x_{8,k} \\ x_{8,k} + \tau x_{9,k} \\ x_{9,k} + \tau A_9^x(\boldsymbol{x}_k) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tau B_5^x(\boldsymbol{x}_k) v_{1,k} \\ \tau B_9^x(\boldsymbol{x}_k) v_{2,k} \end{bmatrix} + \begin{bmatrix} -\tau h_{r,k} \\ 0 \\ 0 \\ 0 \\ 0 \\ -\tau\theta_{c,r,k} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{98}$$

where

$$A_5^x(\boldsymbol{x}_k) = A_4^z(\boldsymbol{z}_k)|_{z=x}, \qquad A_9^x(\boldsymbol{x}_k) = A_7^z(\boldsymbol{z}_k)|_{z=x}$$
$$B_5^x(\boldsymbol{x}_k) = B_4^z(\boldsymbol{z}_k)|_{z=x}, \qquad B_9^x(\boldsymbol{x}_k) = B_7^z(\boldsymbol{z}_k)|_{z=x}$$

Note that the closed-loop dynamics (98) constitute a ninth-order discrete-time system and are driven by the auxiliary controls $v_{1,k}$ and $v_{2,k}$. The output vector equation for the discrete-time system (98) is

$$\boldsymbol{y}_k = \begin{bmatrix} h_k \\ \theta_{c,k} \end{bmatrix} = \begin{bmatrix} x_{2,k} \\ x_{7,k} \end{bmatrix} \tag{99}$$

Careful examination of the system model (98) reveals that its nonlinearities satisfy the matching condition described in the preceding sections. Therefore, the method of discrete-time integral tracking from Sec. VIII can be employed.

### C. Discrete-Time Tracking Control Design

To proceed with the controller design, we let $\boldsymbol{\sigma}_k = [\sigma_{1,k}\ \sigma_{2,k}]^\top = \boldsymbol{S}\boldsymbol{x}_k$. The controllers $v_{1,k}$ and $v_{2,k}$ will be designed to optimize the performance index:

$$J = \sum_{k=k_0}^{\infty} \boldsymbol{\sigma}_k^\top \boldsymbol{\sigma}_k + (\boldsymbol{\sigma}_{k+1} - \boldsymbol{\sigma}_k)^\top \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} (\boldsymbol{\sigma}_{k+1} - \boldsymbol{\sigma}_k) \tag{100}$$

where we select $T_1 = T_2 = 0.98$ to obtain a simple form of the design parameter $\boldsymbol{\Lambda}$ given next. Substituting $\boldsymbol{T} = \text{diag}\{T_1,\ T_2\}$ into the expression for $\boldsymbol{\Lambda}$ given in Proposition 1 yields $\boldsymbol{\Lambda} = \text{diag}\{1/3,\ 1/3\}$. The construction of both $\sigma_{1,k}$ and $\sigma_{2,k}$ ensures that the dynamics of Eq. (98) constrained to the manifold $\{\sigma_1 = 0, \sigma_2 = 0\}$ are asymptotically stable. Using the analysis results of Sec. VIII and the invariant manifold construction algorithm given at the end of Sec. VIII, we proceed with the constructive algorithm by selecting $\boldsymbol{V}$ so that the rank of $[\,\boldsymbol{V} \quad \boldsymbol{B}^x\,]$ is 9. In this case, we select

$$\begin{bmatrix} \boldsymbol{V} & \boldsymbol{B}^x \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where $\boldsymbol{V}^g$ can be directly determined from the first 7 rows of the matrix $[\,\boldsymbol{V} \quad \boldsymbol{B}^x\,]^{-1}$. Next, we determine the matrix $\boldsymbol{F}$ so that the eigenvalues of $\boldsymbol{V}^g\boldsymbol{A}\boldsymbol{V} - \boldsymbol{V}^g\boldsymbol{A}\boldsymbol{B}\boldsymbol{F}$ are at desirable locations. These eigenvalues characterize the closed-loop dynamics restricted to the manifold $\boldsymbol{\sigma} = 0$. In this example, we construct $\sigma_{1,k}$ and $\sigma_{2,k}$ so that the system (98) restricted to 1) $\sigma_1 = 0$ has its discrete-time system poles at

$$\boldsymbol{\lambda}_{\sigma 1} = \{0.9067, \quad 0.9187 \pm \boldsymbol{j}0.0108, \quad 0.9285\}$$

and 2) $\sigma_2 = 0$ has its discrete-time system poles at

$$\boldsymbol{\lambda}_{\sigma 2} = \{0.9492, \quad 0.9126 \pm \boldsymbol{j}0.0133\}$$

Let $\boldsymbol{F}$ be a solution that yields to the preceding eigenvalues. Furthermore, let $\boldsymbol{F}$ have the following structure:

$$\boldsymbol{F} = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & f_{25} & f_{26} & f_{27} \end{bmatrix} \tag{101}$$

where $f_{ij}$ is the $ij$th nonzero entry of the matrix $\boldsymbol{F}$. Hence,

$$\boldsymbol{W} = \boldsymbol{V} - \boldsymbol{B}^x\boldsymbol{F} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -f_{11} & -f_{12} & -f_{13} & -f_{14} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -f_{25} & -f_{26} & -f_{27} \end{bmatrix}$$

and we can deduce from the last step of the constructive algorithm of Sec. VII that $\boldsymbol{S}$ is the last two rows of $[\boldsymbol{W}\ \boldsymbol{B}^x]^{-1}$. Therefore, $\sigma_{1,k}$ and $\sigma_{2,k}$ have the following structure:

$$\begin{bmatrix} \sigma_{1,k} \\ \sigma_{2,k} \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & f_{25} & f_{26} & f_{27} & 1 \end{bmatrix} \boldsymbol{x}_k$$

where $s_{ij}$'s are nonzero entries. Using the matrix $\boldsymbol{F}$ from Eq. (101), we obtain

$$\sigma_{1,k} = x_{5,k} + 3.2728x_{4,k} + 4.0165x_{3,k} + 2.1909x_{2,k} + 0.4481x_{1,k} \tag{102}$$

$$\sigma_{2,k} = x_{9,k} + 2.2557x_{8,k} + 1.6694x_{7,k} + 0.3971x_{6,k}. \tag{103}$$

The control strategy for the discrete-time tracking problem of a discrete-time helicopter model is obtained by substituting Eqs. (102) and (103) into the control strategy (77).

### D. Discussion on Tracking Performance

The helicopter is expected to change its altitude from $h = 0.75$ to 1.25 m. To increase the altitude of a hovering helicopter, one can increase the collective pitch (which increases the lift from each rotor blade) while maintaining constant rotational speed, increase the rotational speed (which increases the lift from each rotor blade while maintaining a constant collective angle), or a combination of the two. To relieve the requirement for excessive rotor blade rotation, we simultaneously change the collective pitch angle from $\theta_c = 0.125$ to 0.2 rad. Simulation results depicting the performance of the constructed discrete-time synergetic controller are presented in Figs. 2–4.

We can see in Fig. 2 that the helicopter altitude follows the prescribed altitude reference. In Fig. 3, we show the time history of the collective blade pitch angle along with the control effort to drive the collective pitch dynamics. Observe that the collective pitch dynamics follow the prescribed reference collective blade pitch. The control strategies developed using the discrete-time synergetic control method also prevent excessive main rotor rotation because, as it is shown in Fig. 4, the rotor rotation does not excessively increase as the helicopter altitude rises. Note that as a result of using the proposed discrete-time control strategies, the helicopter rotor angular velocity decreases as the higher altitude is commanded. Also shown in this figure is the control effort largely responsible for the profile of rotor blade angular velocity.

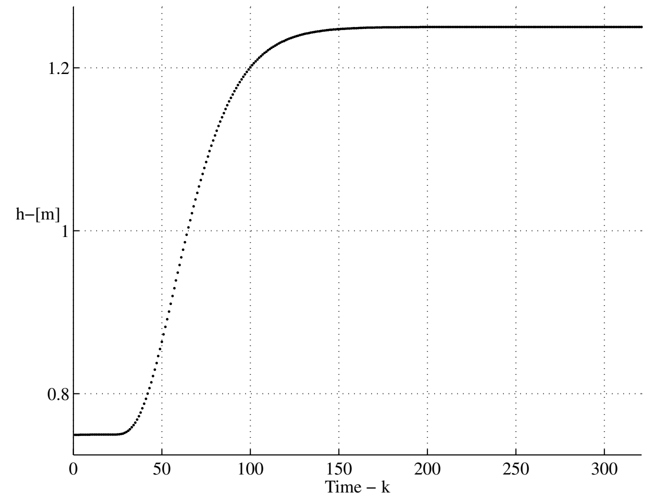


**Fig. 2 A plot of the helicopter altitude $h$ versus time.**

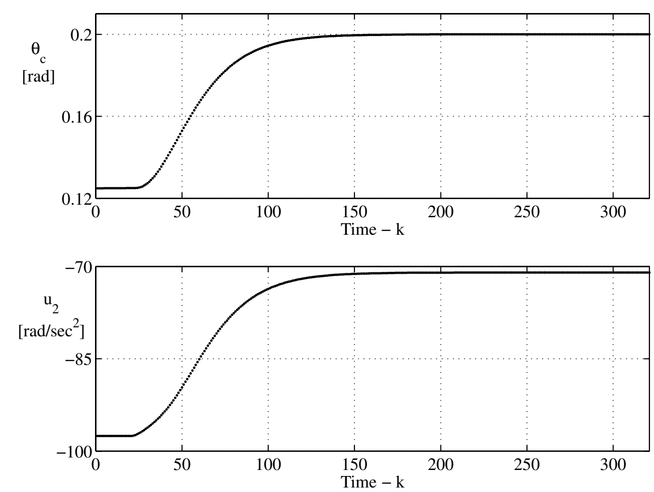


**Fig. 3 Helicopter rotor blade collective pitch control: pitch angle $\theta_c$ and pitch control effort $u_2$.**

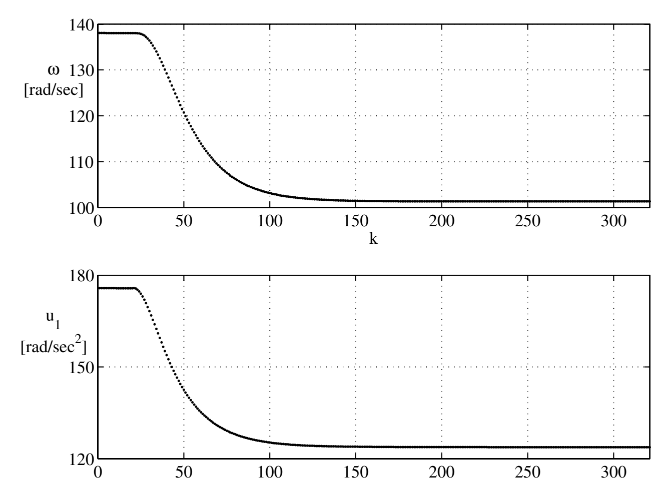


**Fig. 4 A plot of the helicopter rotor blade angular velocity versus time and the control effort $u_1$.**

## X. Conclusions

We proposed novel discrete-time control strategies for solving optimal control problems for a class of nonlinear dynamic systems. The optimal control problem is formulated using a special performance index. Using two different approaches, one employing a discrete-time version of calculus of variations and the other using a dynamic programming approach, we were able to obtain the same optimal control strategy, called the discrete-time synergetic control.

This control law can be derived by solving the associated first-order difference equation for the aggregated variable comprising the controlled system variables. The aggregated variable $\boldsymbol{\sigma}$ must be properly selected so that when the dynamics of the controlled system are confined to the manifold defined by the aggregated variable, the resulting reduced-order dynamics are stable.

We established connections between the synergetic control approach and a version of the discrete-time variable-structure sliding-mode control method. We showed that the first-order difference equation for control law derivation in the synergetic method corresponds to the reaching conditions of the discrete-time variable-structure sliding-mode control. Moreover, we showed that the difference equation used to derive the control law of discrete-time synergetic control is the same as the difference equation for the linear reaching law of a variable-structure sliding-mode control using the $\beta$-equivalent-control approach.

In addition, synergetic control was shown to provide the same controller as the LQR, with a special performance index for the case of linear time-invariant dynamic systems. We provided the closed-loop stability analysis for the case when the nonlinear plant model contained matched nonlinearities. We showed that the closed-loop nonlinear system stability is determined by the stability of the first-order difference equation, used to derive the discrete-time synergetic optimizing control law, and the stability of the controlled system confined to the manifold defined by the aggregated variable. We also presented a constructive algorithm that generates an invariant manifold such that the closed-loop nonlinear system driven by the discrete-time optimizing synergetic controller is asymptotically stable.

Next, we offered a method for constructing the discrete-time synergetic optimal control for the purpose of tracking piecewise constant reference applied to nonlinear plants with matched nonlinearities. The strategy employs an integral action and achieves asymptotically zero-error tracking performance.

The results obtained are illustrated with a numerical example involving an application of the proposed method to optimal control of a highly nonlinear helicopter model yielding excellent closed-loop performance.

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